Lecture Note on Elementary Differential Geometry

Ling-Wei Luo*

Institute of Physics, Academia Sinica

July 20, 2019

Abstract

This is a note based on a course of elementary differential geometry as I gave the lectures in the *NCTU-Yau Journal Club: Interplay of Physics and Geometry* at Department of Electrophysics in National Chiao Tung University (NCTU) in Spring semester 2017. The contents of remarks, supplements and examples are highlighted in the red, green and blue frame boxes respectively. The supplements can be omitted at first reading. The basic knowledge of the *differential forms* can be found in the lecture notes given by Dr. Sheng-Hong Lai (NCTU) and Prof. Jen-Chi Lee (NCTU) on the website. The website address of *Interplay of Physics and Geometry* is `http://web.it.nctu.edu.tw/~string/journalclub.htm` or `http://web.it.nctu.edu.tw/~string/ipg/`.

Contents

1 Curve on $\mathbb{E}^2$

We define n-dimensional Euclidean space $\mathbb{E}^n$ as a n-dimensional real space $\mathbb{R}^n$ equipped a dot product defined n-dimensional vector space.

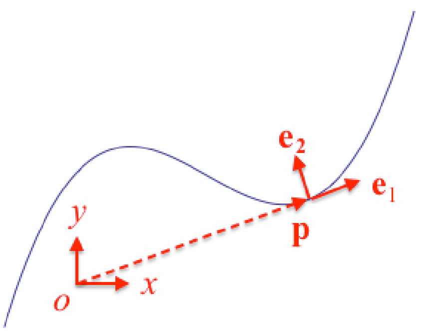


Figure 1: A curve.

Tangent vector In 2-dimensional Euclidean space, an $\mathbb{E}^2$ plane, we parametrize a curve $\mathbf{p}(t) = \big(x(t), y(t)\big)$ by one parameter t with respect to a reference point $\boldsymbol{o}$ with a *fixed* Cartesian coordinate frame. The *velocity* vector at point $\mathbf{p}$ is given by $\dot{\mathbf{p}}(t) = \big(\dot{x}(t), \dot{y}(t)\big)$ with the *norm*

$$|\dot{\mathbf{p}}(t)| = \sqrt{\dot{\mathbf{p}} \cdot \dot{\mathbf{p}}} = \sqrt{\dot{x}^2 + \dot{y}^2}\,, \qquad (1)$$

*Electronic address: lwluo@gate.sinica.edu.tw

where $\dot{x} := dx/dt$. The arc length s in the interval $[a, b]$ can be calculated by

$$s = \int ds = \int \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \int_a^b |\dot{\mathbf{p}}(t)|\, dt\,. \tag{2}$$

The arc length can be a function of parameter t given by

$$s(t) = \int_a^t |\dot{\mathbf{p}}(t')|\, dt'\,. \tag{3}$$

From the *fundamental theorem of calculus*, we have

$$\left|\frac{ds}{dt}\right| \neq 0 \quad \Longrightarrow \quad \dot{s}(t) = |\dot{\mathbf{p}}(t)| > 0\,. \tag{4}$$

According to the *inverse function theorem*, we have $t = t(s)$. One can parametrize the curve by arc length s as $\mathbf{p}(s) = \big(x(s), y(s)\big)$. The corresponding velocity vector should be $\mathbf{p}'(s) = \big(x'(s), y'(s)\big)$, where we have $x' := dx/ds$. We can rewrite the derivatives of x and y with respect to s as

$$\begin{cases} x' = \dfrac{dx}{ds} = \dfrac{dx}{dt}\dfrac{dt}{ds} = \dot{x}\dfrac{dt}{ds}\,, \\[2ex] y' = \dot{y}\dfrac{dt}{ds}\,. \end{cases} \tag{5}$$

Thus, the norm of the velocity vector parametrized by s can be calculated as

$$|\mathbf{p}'(s)| = \sqrt{x'^2 + y'^2} = \sqrt{\dot{x}^2 + \dot{y}^2}\frac{dt}{ds} = |\dot{\mathbf{p}}|\frac{dt}{ds} = \frac{ds}{dt}\frac{dt}{ds} = 1\,, \tag{6}$$

which implies that the velocity vector $\mathbf{p}'(s)$ is a *unit* vector. We can define a *unit tangent vector* as a velocity vector parametrized by s

$$\boxed{\boldsymbol{T} \equiv \mathbf{e}_1 := \mathbf{p}'(s)\,.} \tag{7}$$

Normal vector Due to $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{p}' \cdot \mathbf{p}' = 1$, we have

$$\mathbf{e}_1' \cdot \mathbf{e}_1 + \mathbf{e}_1 \cdot \mathbf{e}_1' = 0 \quad \Longrightarrow \quad \mathbf{e}_1' \cdot \mathbf{e}_1 = 0 \quad \Longrightarrow \quad \mathbf{e}_1' \perp \mathbf{e}_1, \tag{8}$$

it indicates that $\mathbf{e}_1'$ is a normal vector. The *principle normal vector* is defined by

$$\boxed{\boldsymbol{N} \equiv \mathbf{e}_2 := \frac{\mathbf{e}_1'}{|\mathbf{e}_1'|}} \tag{9}$$

as a unit normal vector at $\mathbf{p}(s)$. The curvature of a curve $\mathbf{p}(s)$ is given by $\kappa(s) = |\mathbf{e}_1'(s)| > 0$, which can be realized as a norm of the *acceleration* vector $\boldsymbol{a} := \mathbf{e}_1' = \mathbf{p}''$. Therefore, we have a relation

$$\boxed{\mathbf{e}_1' = \kappa(s)\mathbf{e}_2\,.} \tag{10}$$

Remark. If a vector V is an *unit* vector, $|V| = 1$, the corresponding derivative vector would be perpendicular to itself, *i.e*.

$$V' \perp V\,. \tag{11}$$

Osculating plane The plane is spanned by the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ is called *osculating plane*.

Newton's second law In classical physics, we have a momentum vector $\boldsymbol{p} = m\boldsymbol{T} = m\mathbf{p}'$ with mass m. The force $\boldsymbol{F}$ is defined by Newton's second law

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{ds} = m\frac{d\boldsymbol{T}}{ds} = m\boldsymbol{a} = m\mathbf{p}'' \tag{12}$$

with respect to parameter s.

Frame A set of vector $\mathbf{e}_1, \mathbf{e}_2$ equipped with a point $\mathbf{p}$ calls frame. In such of case, a frame at $\mathbf{p}$ is denoted by $(\mathbf{p}; \mathbf{e}_1, \mathbf{e}_2)$.

Frenet-Serret formula in 2D From the orthonormality condition $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ $(i, j = 1, 2)$, we have

$$\mathbf{e}_i' \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_j' = 0 \tag{13a}$$

$$\implies \mathbf{e}_1' \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}_2' = \kappa + \mathbf{e}_1 \cdot \mathbf{e}_2' = 0 \tag{13b}$$

$$\implies \mathbf{e}_1 \cdot \mathbf{e}_2' = -\kappa \quad (\mathbf{e}_2' \text{ has component } -\kappa \text{ along } \mathbf{e}_1 \text{ direction}) \tag{13c}$$

$$\implies \boxed{\mathbf{e}_2' = -\kappa\mathbf{e}_1\,.} \tag{13d}$$

As a result, we have the following relations

$$\begin{cases} \mathbf{p}' = & +\mathbf{e}_1 & \\ \mathbf{e}_1' = & & +\kappa\mathbf{e}_2 \\ \mathbf{e}_2' = & -\kappa\mathbf{e}_1 & \end{cases} \implies \begin{pmatrix} \mathbf{p}' \\ \mathbf{e}_1' \\ \mathbf{e}_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \tag{14}$$

called *Frenet-Serret formula*.

Example (Circle in $\mathbb{E}^2$)**.** A circle with radius r can be parametrized by $\mathbf{p}(t) = (r\cos t, r\sin t)$ with $0 \leq t \leq 2\pi$.

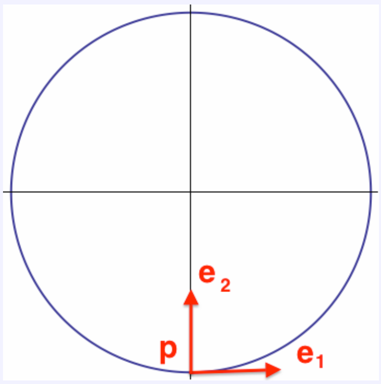


Figure 2: A circle.

The tangent vector is

$$\dot{\mathbf{p}}(t) = (-r\sin t, r\cos t) \tag{15}$$

with norm

$$|\dot{\mathbf{p}}| = \sqrt{r^2\sin^2 t, r^2\cos^2 t)} := r\,. \tag{16}$$

The arc length $s(t)$ is

$$s(t) = \int_0^t |\mathbf{p}(t')|dt' = \int_0^t r\,dt' = rt\,. \tag{17}$$

Therefore, the circumference is

$$L = \int_0^{2\pi} |\mathbf{p}(t')|dt' = \int_0^{2\pi} r\,dt' = 2\pi r\,. \tag{18}$$

By $t = s/r$, the circle $\mathbf{p}(s)$ and its tangent vector are

$$\mathbf{p}(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}\right) \tag{19a}$$

and

$$\mathbf{p}'(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right) = \mathbf{e}_1 = \boldsymbol{T} \tag{19b}$$

respectively. From (19b), we have

$$\mathbf{e}_1'(s) = \left(-\frac{1}{r}\cos\frac{s}{r}, -\frac{1}{r}\sin\frac{s}{r}\right). \tag{20}$$

The curvature κ can be obtained by

$$\kappa = |\mathbf{e}_1'| = \sqrt{\frac{1}{r^2}\cos^2\frac{s}{r} + \frac{1}{r^2}\sin^2\frac{s}{r}} = \frac{1}{r}\,, \tag{21}$$

which is the inverse of the constant radius r. The normal vector can be calculated by

$$\mathbf{e}_2 = \frac{\mathbf{e}_1'}{|\mathbf{e}_1'|} = r\left(-\frac{1}{r}\cos\frac{s}{r} - \frac{1}{r}\sin\frac{s}{r}\right) = \left(-\cos\frac{s}{r}, \sin\frac{s}{r}\right). \tag{22}$$

Gauss map Gauss map $\mathbf{G}$ is a mapping which globally send all the points $\mathbf{p}$ of curve to a *unit circle* S^1 (a Gauss circle) centered at $\mathbf{c}$ and send the corresponding normal vector $\mathbf{e}_2$ to a radius vector from $\mathbf{c}$ pointing to S^1, which is shown as Fig. 3. Therefore, $\mathbf{e}_2$ can be represented as a point on S^1.

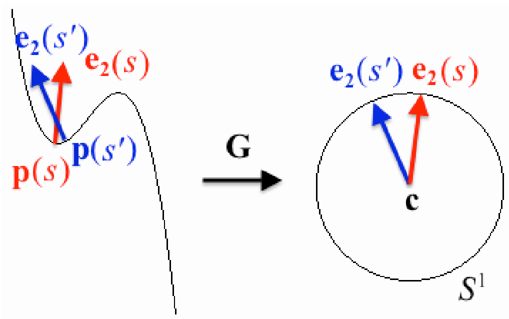


Figure 3: The Gauss map $\mathbf{G}$.

Let's consider two normal vectors $\mathbf{e}_2(s)$ and $\mathbf{e}_2(s')$ with respect to two infinitesimal points $\mathbf{p}(s)$ and $\mathbf{p}(s')$, where $s' = s + \Delta s$ is infinitesimal close to s. We can expand $\mathbf{e}_2(s')$ at s:

$$\begin{aligned}\mathbf{e}_2(s') &= \mathbf{e}_2(s + \Delta s)\\ &\approx \mathbf{e}_2(s) + \mathbf{e}_2'(s)\Delta s\\ &= \mathbf{e}_2(s) + (-\kappa(s)\mathbf{e}_1(s))\Delta s\\ &= \mathbf{e}_2(s) + (-\kappa(s)\Delta s)\mathbf{e}_1(s)\,,\end{aligned} \tag{23}$$

which is the parametrization of a point under the Gauss map. Thus, we know the distance between two infinitesimal point $\mathbf{e}_2(s)$ and $\mathbf{e}_2(s')$ on Gauss circle given by

$$|\mathbf{e}_2(s') - \mathbf{e}_2(s)| = |\Delta \mathbf{e}_2| = \kappa(s)\Delta s\,. \tag{24}$$

And we also have

$$\Delta \mathbf{p} \equiv \mathbf{p}(s') - \mathbf{p}(s) \approx \big(\mathbf{p}(s) + \mathbf{p}'(s)\Delta s\big) - \mathbf{p}(s) = \mathbf{p}'(s)\Delta s \qquad \Longrightarrow \qquad |\mathbf{p}(s') - \mathbf{p}(s)| = |\Delta \mathbf{p}| = \Delta s\,. \tag{25}$$

Therefore, in the *local* region, the ratio of the length between two points on the Gauss circle and curve, *i.e.*, $|\Delta \mathbf{e}_2|/|\Delta \mathbf{p}|$ can be calculate by

$$\frac{|\mathbf{e}_2(s') - \mathbf{e}_2(s)|}{|\mathbf{p}(s') - \mathbf{p}(s)|} = \frac{|\Delta \mathbf{e}_2|}{|\Delta \mathbf{p}|} = \frac{\kappa(s)\Delta s}{\Delta s} = \kappa(s)\,, \tag{26}$$

which measure the curvature of a curve, $\kappa(s)$, at the neighborhood of a local point $\mathbf{p}$.

According to the example of circle, we assume a vector $\mathbf{q} = \mathbf{p} + r\mathbf{e}_2 = \mathbf{p} + (1/\kappa)\mathbf{e}_2$. The derivative of $\mathbf{q}$ is

$$\mathbf{q}' = \mathbf{p}' + \frac{1}{\kappa}\mathbf{e}_2' = \mathbf{e}_1 + \frac{1}{\kappa}(-\kappa \mathbf{e}_1) = 0\,, \tag{27}$$

which means that $\mathbf{q}$ is fixed, *i.e.*, $\mathbf{q}$ is the center of the osculating circle with radius $1/\kappa$. By considering the Gauss map of a circle. The radius vector should be $\mathbf{e}_2$ and the center $\mathbf{c}$ of Gauss circle corresponds to the point $\mathbf{q}$ of the osculating circle which is the circle itself. Thus, the Gauss circle can be imaged by rescaling the radius of osculating circle to unity.

Example (Curvature of ellipse)**.** An ellipse is described by $\mathbf{p}(t) = \big(x(t), y(t)\big)$ with the parametrization of the coordinates $x(t) = a\cos t$ and $y(t) = b\sin t$ $(a > b > 0)$, *i.e.*,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1\,. \tag{28}$$

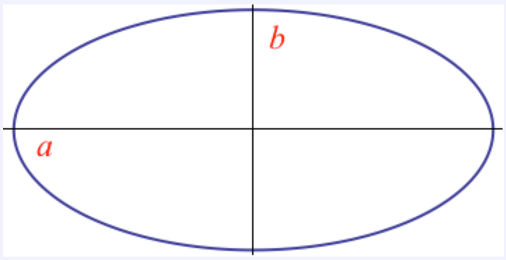


Figure 4: An ellipse.

The tangent vector is

$$\dot{\mathbf{p}}(t) = \big(-a\sin t, b\cos t\big) \tag{29}$$

By changing the parameter to s, we have to calculate ds/dt first:

$$\frac{ds}{dt} = |\dot{\mathbf{p}}| = \sqrt{a^2\sin^2 t + b^2\cos^2 t} \qquad \Longrightarrow \qquad \frac{dt}{ds} = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}} = \frac{1}{\dot{s}}\,. \tag{30}$$

Therefore, the tangent vector parametrized by s is obtained by

$$\begin{aligned} \mathbf{e}_1 = \mathbf{p}' &= \frac{d\mathbf{p}}{dt}\frac{dt}{ds} \\ &= \left(\frac{-a\sin t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}, \frac{b\cos t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}\right) = \left(\frac{-a\sin t}{\dot{s}}, \frac{b\cos t}{\dot{s}}\right), \end{aligned} \tag{31}$$

Subsequently, we have

$$\mathbf{e}_2 = \left(\frac{-b\cos t}{\dot{s}}, \frac{-a\sin t}{\dot{s}} \right). \tag{32}$$

However

$$\mathbf{e}_1' = \frac{d\mathbf{e}_1}{ds} = \frac{d\mathbf{e}_1}{dt}\frac{dt}{ds} = \kappa \mathbf{e}_2 \,. \tag{33}$$

As a result, the curvature is

$$\kappa(t) = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} \,. \tag{34}$$

If we consider the particular case of $a = b$, an ellipse reduce to a circle with curvature $\kappa = 1/a$.

2 Curve in $\mathbb{E}^3$

In $\mathbb{E}^3$, a curve is parametrized as $\mathbf{p}(t) = \big(x(t), y(t), z(t)\big)$ and we have to look for an orthonormal frame at p denoted by $(\mathbf{p}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ The vector $\mathbf{e}_1 = \mathbf{p}'$ is uniquely defined by the same way. Due to $\mathbf{e}_1' \perp \mathbf{e}_1$, vector $\mathbf{e}_1'$ should be proportional to $\mathbf{e}_2$ or $\mathbf{e}_3$. Now we can fix $\mathbf{e}_1' = \kappa \mathbf{e}_2$ as the previous section.

Binormal vector Now we define a unit vector orthogonal to $\boldsymbol{T}$ and $\boldsymbol{N}$ called *binormal* vector

$$\boxed{\begin{aligned} \boldsymbol{B} &:= \boldsymbol{T} \wedge \boldsymbol{N} \\ &\equiv \mathbf{e}_1 \wedge \mathbf{e}_2 := \mathbf{e}_3 \,, \end{aligned}} \tag{35}$$

where $\wedge$ is the *exterior product* or *wedge product*.

Remark. In 3-dimensional space, the *exterior product* $\wedge$ is the same to the usual *cross product* $\times$ of two vectors.

By orthonormality condition $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ $(i, j = 1, 2, 3)$, we have

$$\mathbf{e}_i' \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_j' = 0 \,, \tag{36}$$

which implies:
(i) If $i = j$, we have $\mathbf{e}_i' \perp \mathbf{e}_i$, $\mathbf{e}_2'$ should be the combination of $\mathbf{e}_1$ and $\mathbf{e}_3$.
(ii) If $i \neq j$, we have

$$\begin{cases} 0 = \mathbf{e}_1' \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}_2' = (\kappa \mathbf{e}_2) \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}_2' = \kappa + \mathbf{e}_1 \cdot \mathbf{e}_2' \quad (i = 1, j = 2) \,, & \text{(37a)} \\ 0 = \mathbf{e}_1' \cdot \mathbf{e}_3 + \mathbf{e}_1 \cdot \mathbf{e}_3' = (\kappa \mathbf{e}_2) \cdot \mathbf{e}_3 + \mathbf{e}_1 \cdot \mathbf{e}_3' = 0 + \mathbf{e}_1 \cdot \mathbf{e}_3' \quad (i = 1, j = 3) \,. & \text{(37b)} \end{cases}$$

Therefore, with the result (37a), we have to assume that

$$\boxed{\mathbf{e}_2' = -\kappa(s)\mathbf{e}_1 + \tau(s)\mathbf{e}_3 \,.} \tag{38}$$

By comparing to (13d), it contains an additional term related to $\mathbf{e}_3$. For $i = 2, j = 3$, we obtain

$$0 = \mathbf{e}_2' \cdot \mathbf{e}_3 + \mathbf{e}_2 \cdot \mathbf{e}_3' = (-\kappa \mathbf{e}_1 + \tau \mathbf{e}_3) \cdot \mathbf{e}_3 + \mathbf{e}_2 \cdot \mathbf{e}_3' = \tau + \mathbf{e}_2 \cdot \mathbf{e}_3' \,. \tag{39}$$

Due to (i) and (37b), $\mathbf{e}_3'$ should be perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_3$. As a result, we obtain the unique solution that

$$\boxed{\mathbf{e}_3' = -\tau\mathbf{e}_2\,,} \tag{40}$$

where $\tau(s)$ is called *torsion* of a curve $\mathbf{p}(s)$. The geometric meaning of torsion is that it make the point of the curve leave for the osculating plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$.

Remark. Apparently, the torsion of a curve is always related to the binormal vector $\boldsymbol{B} \equiv \mathbf{e}_3$.

Frenet-Serret formula in 3D As a result, we have Frenet-Serret formula:

$$\begin{cases} \mathbf{p}' = & +\mathbf{e}_1 \\ \mathbf{e}_1' = & & +\kappa\mathbf{e}_2 \\ \mathbf{e}_2' = & -\kappa\mathbf{e}_1 & & +\tau\mathbf{e}_3 \\ \mathbf{e}_3' = & & -\tau\mathbf{e}_2 \end{cases} \implies \begin{pmatrix} \mathbf{p}' \\ \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \tag{41}$$

Remark. If one defines $\boldsymbol{B} := \boldsymbol{N} \wedge \boldsymbol{T}$, then one should assume $\mathbf{e}_2' = -\kappa(s)\mathbf{e}_1 - \tau(s)\mathbf{e}_3$ and obtain $\mathbf{e}_3' = +\tau\mathbf{e}_2$.

Parametrization of a curve in a neighborhood of s_0 One can do the Taylor expansion of $\mathbf{p}(s)$ at s_0.

- First order:

$$\mathbf{p}(s) \approx \mathbf{p}(s_0) + \left.\frac{d\mathbf{p}}{ds}\right|_{s=s_0}(s-s_0) = \mathbf{p}(s_0) + \mathbf{e}_1(s_0)(s-s_0)\,. \tag{42}$$

- Second order:

$$\begin{aligned} \mathbf{p}(s) &\approx \mathbf{p}(s_0) + \mathbf{p}'(s_0)(s-s_0) + \frac{1}{2!}\mathbf{p}''(s_0)(s-s_0)^2 \\ &= \mathbf{p}(s_0) + \mathbf{e}_1(s_0)(s-s_0) + \frac{1}{2}\kappa(s_0)\mathbf{e}_2(s_0)(s-s_0)^2\,. \end{aligned} \tag{43}$$

- Third order:

$$\begin{aligned} \mathbf{p}(s) &\approx \mathbf{p}(s_0) + \mathbf{p}'(s_0)(s-s_0) + \frac{1}{2!}\mathbf{p}''(s_0)(s-s_0)^2 + \frac{1}{3!}\underbrace{\mathbf{p}'''(s_0)}_{\mathbf{p}'''=(\mathbf{p}'')'=\kappa'\mathbf{e}_2+\kappa\mathbf{e}_2'=\kappa'\mathbf{e}_2+\kappa(-\kappa\mathbf{e}_1+\tau\mathbf{e}_3)}(s-s_0)^3 \\ &= \mathbf{p}(s_0) + \mathbf{e}_1(s_0)(s-s_0) + \frac{1}{2}\kappa(s_0)\mathbf{e}_2(s_0)(s-s_0)^2 \\ &\quad + \frac{1}{6}\Big(-\kappa^2(s_0)\mathbf{e}_1(s_0) + \kappa'(s_0)\mathbf{e}_2(s_0) + \underbrace{\kappa(s_0)\tau(s_0)\mathbf{e}_3(s_0)}_{\text{leading term}}\Big)(s-s_0)^3\,. \end{aligned} \tag{44}$$

We only consider the leading term in the third order expansion, then we have

$$\mathbf{p}(s) \approx \mathbf{p}(s_0) + \mathbf{e}_1(s_0)(s-s_0) + \frac{1}{2}\kappa(s)\mathbf{e}_2(s_0)(s-s_0)^2 + \frac{1}{6}\big(\kappa(s_0)\tau(s_0)\mathbf{e}_3(s_0)\big)(s-s_0)^3\,. \tag{45}$$

Example (Helix in $\mathbb{E}^3$)**.** A helix is parametrized as $\mathbf{p} = \big(x(t), y(y), z(t)\big)$ with

$$\begin{cases} x(t) = a\cos t\,, \\ y(t) = a\sin t\,, \\ z(t) = bt\,. \end{cases} \tag{46}$$

The tangent vector and the corresponding norm are

$$\dot{\mathbf{p}} = \big(\dot{x}, \dot{y}, \dot{z}\big) = \big(-a\sin t, a\cos t, b\big) \tag{47}$$

and

$$|\dot{\mathbf{p}}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2} = \dot{s}\,. \tag{48}$$

The relation of s and t can be obtained by

$$s(t) = \int_0^t \frac{ds}{dt'}\,dt' = \int_0^t \sqrt{a^2+b^2}\,dt' := ct \qquad \Longrightarrow \qquad t = \frac{s}{c}\,. \tag{49}$$

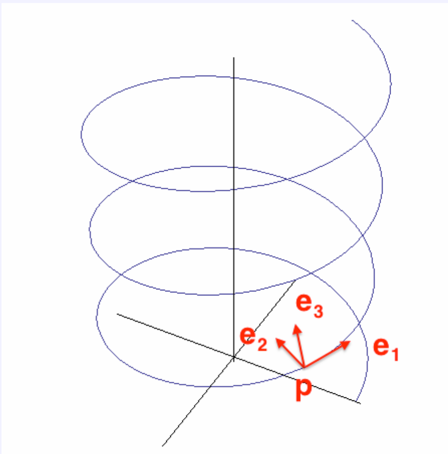


Figure 5: A helix.

Subsequently, we have tangent vector

$$\mathbf{e}_1 = \mathbf{p}' = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right) \tag{50}$$

and

$$\mathbf{e}_1' = \mathbf{p}'' = \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right). \tag{51}$$

So the curvature is

$$\kappa = |\mathbf{e}_1'| = \frac{a}{c^2} = \frac{a}{a^2+b^2}\,. \tag{52}$$

and the normal vector can be obtained by

$$\mathbf{e}_1' = \kappa\mathbf{e}_2 \qquad \Longrightarrow \qquad \mathbf{e}_2 = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right). \tag{53}$$

Finally, we have binormal vector

$$\mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 = \left(\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right). \tag{54}$$

Due to

$$\mathbf{e}_3' = \left(\frac{b}{c^2}\cos\frac{s}{c}, \frac{b}{c^2}\sin\frac{s}{c}, 0\right), \tag{55}$$

we can calculate the torsion of $\mathbf{p}(s)$ form (40):

$$\tau = \frac{b}{c^2} = \frac{b}{a^2+b^2}. \tag{56}$$

3 Surface theory in $\mathbb{E}^3$

We consider a 2-dimensional surface $\mathcal{M}$ in $\mathbb{E}^3$, we parametrize the surface by *two* variables u and v written as $\mathbf{p}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$.

Remark. If the point $\mathbf{p}(u,v)$ moves along u direction, *i.e.*, parametrized by u only, we call the trajectory u-curve. The infinitesimal vector along u is

$$\Delta\mathbf{p}|_u = \mathbf{p}(u+\Delta u, v) - \mathbf{p}(u,v) \approx \mathbf{p}(u,v) + \frac{\partial \mathbf{p}(u,v)}{\partial u}\Delta u - \mathbf{p}(u,v) = \mathbf{p}_u \Delta u. \tag{57}$$

Similarly, we have v-curve along v direction and

$$\Delta\mathbf{p}|_v \approx \mathbf{p}_v \Delta v. \tag{58}$$

Therefore, we have

$$\Delta\mathbf{p} \approx \mathbf{p}_u \Delta u + \mathbf{p}_v \Delta v. \tag{59}$$

Tangent vector The differential of $\mathbf{p}$ is

$$d\mathbf{p} = (dx, dy, dz) \tag{60}$$

with

$$\begin{cases} dx = \dfrac{\partial x}{\partial u}du + \dfrac{\partial x}{\partial v}dv = x_u du + x_v dv, \\ dy = \dfrac{\partial y}{\partial u}du + \dfrac{\partial y}{\partial v}dv = y_u du + y_v dv, \\ dz = \dfrac{\partial z}{\partial u}du + \dfrac{\partial z}{\partial v}dv = z_u du + z_v dv, \end{cases} \tag{61}$$

where $x_u := \partial x/\partial u$. Therefore, we can write $d\mathbf{p}$ as

$$d\mathbf{p} = \frac{\partial \mathbf{p}}{\partial u}du + \frac{\partial \mathbf{p}}{\partial v}dv := \mathbf{p}_u du + \mathbf{p}_v dv, \tag{62}$$

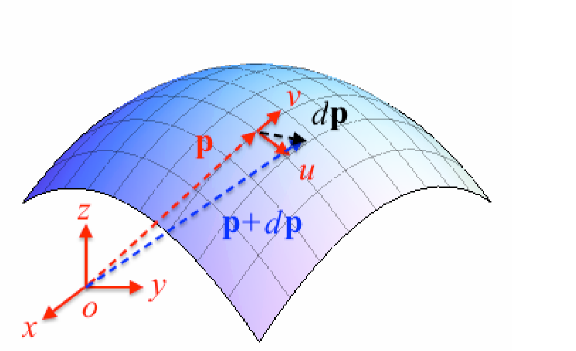


Figure 6: A surface.

where

$$\begin{cases} \mathbf{p}_u := (x_u, y_u, z_u) \\ \mathbf{p}_v := (x_v, y_v, z_v) \end{cases} \tag{63}$$

are called velocity vectors along u and v respectively.

Remark. The vector $\mathbf{p}$ in $\mathbb{E}^3$ in the *Cartesian coordinate system* can be written as

$$\mathbf{p} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} := x^a \delta_a \quad (a = 1, 2, 3)\,, \tag{64}$$

where $\{\delta_a\}$ is a *fixed* reference frame of $\mathbb{E}^3$. So that we have differential

$$d\mathbf{p} = (dx^a)\delta_a + x^a(d\delta_a)\,. \tag{65}$$

Because δ_a is fixed, *i.e.* $d\delta_a = 0$, it leads to the differential of $\mathbf{p}$

$$d\mathbf{p} = (dx^a)\delta_a = (dx, dy, dz)\,. \tag{66}$$

The general situation for *non-fixed* frame in space $\mathcal{M}^n$ will be discussed in the Sec. 4 of moving frame.

Tangent space We call a space spanned by $\mathbf{p}_u$ and $\mathbf{p}_v$ at point $\mathbf{p}$ a tangent space denoted by $T_{\mathbf{p}}\mathcal{M}$.

Supplement (Tangent bundle)**.** In tangent space with dimension 2, a vector V has a generalized coordinate transformation $GL(2;\mathbb{R})$, which is $\tilde{u}^i = \tilde{u}^i(u)$ and gives the transformation for vector

$$V = V^i \mathbf{p}_i = \widetilde{V}^j \tilde{\mathbf{p}}_j\,. \tag{67a}$$

The transformation of the basis and components are given by

$$\tilde{\mathbf{p}}_j(\tilde{u}) = \frac{\partial u^i}{\partial \tilde{u}^j}\mathbf{p}_i(u) \quad (\textit{Pushforward})\,, \tag{67b}$$

$$V^i(u) = \widetilde{V}^j(\tilde{u})\frac{\partial u^i}{\partial \tilde{u}^j} \quad (\textit{Pullback})\,, \tag{67c}$$

where

$$(\mathbf{J})^i{}_j := \frac{\partial u^i}{\partial \tilde{u}^j} \tag{68}$$

is an element of the Jacobian matrix $\mathbf{J}$ of the general linear transformation $GL(2;\mathbb{R})$. The map *pushforward* (*pullback*) means that the *covariant* (*contravariant*) quantities expressed in new (old) coordinate system under the generalized coordinate transformation from old (new) coordinate system.

We can collect all pairs of the points $\mathbf{p}$ on $\mathcal{M}$ and their corresponding tangent space $T_{\mathbf{p}}\mathcal{M}$. A *tangent bundle* $T\mathcal{M}$ is defined by the collection of $T_{\mathbf{p}}\mathcal{M}$, *i.e.*,

$$T\mathcal{M} = \bigcup_{\mathbf{p}\in\mathcal{M}} T_{\mathbf{p}}\mathcal{M}\,. \tag{69}$$

A *tangent bundle* $T\mathcal{M}$ is a *vector bundle* denoted by $(E, \mathcal{M}, \pi)$, which is a special *fibre bundle* with

- *base space* B: $\mathcal{M}$;
- *standard (typical) fibre* F over **p** (an object defined at **p**): $T_{\mathbf{p}}\mathcal{M}$;
- *total space* E: a collection of all $T_{\mathbf{p}}\mathcal{M}$;
- *bundle projection* π (an element **u** of bundle is projected by the fibre to the corresponding point **p**): $\pi(\mathbf{u}) = \mathbf{p}$ for $\mathbf{u} \in T\mathcal{M}$;
- *structure group* G: $GL(2;\mathbb{R})$;
- *transition function* $t^i{}_j$: Jacobian matrix **J** of $GL(2;\mathbb{R})$,

and we call $E_{\mathbf{p}} = \pi^{-1}(\mathbf{p})$ the fibre of E over point **p**.

First fundamental (quadratic) form we define

$$\begin{aligned}\mathbf{I} &:= d\mathbf{p}\cdot d\mathbf{p} && (70a)\\ &= \underbrace{\mathbf{p}_u\cdot\mathbf{p}_u}_{E}\,dudu + 2\,\underbrace{\mathbf{p}_u\cdot\mathbf{p}_v}_{F}\,dudv + \underbrace{\mathbf{p}_v\cdot\mathbf{p}_v}_{G}\,dvdv\\ &= E\,dudu + 2F\,dudv + G\,dudv && (70b)\\ &= \begin{pmatrix} du & dv\end{pmatrix}\begin{pmatrix} E & F\\ F & G\end{pmatrix}\begin{pmatrix} du\\ dv\end{pmatrix}. && (70c)\end{aligned}$$

called the *first fundamental form* or *metric tensor* of surface $\mathcal{M}$, which is a *symmetric* quadratic form rather than an *exterior* 2-form.

Remark. In the case of $F = 0$, the first fundamental form is

$$\begin{aligned}\mathbf{I} &= E\,dudu + G\,dvdv && (71a)\\ &= \begin{pmatrix} du & dv\end{pmatrix}\begin{pmatrix} E & 0\\ 0 & G\end{pmatrix}\begin{pmatrix} du\\ dv\end{pmatrix}. && (71b)\end{aligned}$$

In such case, we call (u, v) an *isothermal coordinates* if $E = G$. Therefore, the component of metric is

$$g_{ij} = E\,\delta_{ij} \quad \text{with} \quad E = \mathbf{p}_i\cdot\mathbf{p}_i = |\mathbf{p}_i|^2 > 0\,, \tag{72}$$

and we say that g_{ij} is *conformally equivalent* to δ_{ij}, which preserved the angle between any two vectors. Because δ_{ij} gives the flat space, we say that g_{ij} is *conformally flat*.

We consider a curve on the surface, that means u and v should be parametrized by one variable t, *i.e.*, $u = u(t)$ and $v = v(t)$. The curve $\mathbf{p}(t) = \mathbf{p}\big(u(t), v(t)\big)$. The tangent vector is obtained by

$$\dot{\mathbf{p}} = \frac{\partial\mathbf{p}}{\partial u}\frac{du}{dt} + \frac{\partial\mathbf{p}}{\partial v}\frac{dv}{dt} = \mathbf{p}_u\dot{u} + \mathbf{p}_v\dot{v}\,, \tag{73}$$

and the corresponding norm is

$$|\dot{\mathbf{p}}| = \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\,. \tag{74}$$

We would like to calculate the arc length of a curve by

$$s = \int ds = \int |\dot{\mathbf{p}}| \, dt \tag{75}$$

$$= \int \underbrace{\sqrt{Edu^2 + 2Fdudv + Gdv^2}}_{\sqrt{ds^2}} \tag{76}$$

$$= \int \underbrace{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}}_{1} \, ds = \int |\mathbf{p}'| \, ds \, . \tag{77}$$

Therefore, we have

$$\boxed{|\mathbf{p}'| = 1 \, .} \tag{78}$$

We would alwalys write the first fundamental form with $u = u^1$ and $v = u^2$ as

$$\boxed{\mathbf{I} \equiv ds^2 \equiv g = g_{ij} du^i du^j \quad (i, j = 1, 2) \, ,} \tag{79}$$

where

$$\boxed{g_{ij} = \mathbf{p}_i \cdot \mathbf{p}_j \longrightarrow \begin{pmatrix} E & F \\ F & G \end{pmatrix}} \tag{80}$$

is the metric tensor represented as a 2×2 matrix on the surface $\mathcal{M}$. The inverse of g_{ij} is defined by

$$\boxed{g^{ki} g_{ij} = \delta^k_j \, .} \tag{81}$$

Remark. The first fundamental form describes the distance of two points on the surface $\mathcal{M}$, which gives the *intrinsic* structure of $\mathcal{M}$.

Supplement (Induced metric)**.** We can regard $\mathbf{p}$ as a set of functions defined on the surface $\mathcal{M}$, the differential of $\mathbf{p}$ is actually an *infinitesimal tangent vector* laid on $\mathcal{M}$

$$d\mathbf{p} = \frac{\partial \mathbf{p}}{\partial u} du + \frac{\partial \mathbf{p}}{\partial v} dv = \left(du \frac{\partial}{\partial u} + dv \frac{\partial}{\partial u} \right) \mathbf{p} = (du^i \partial_i) \mathbf{p} \, , \tag{82}$$

which can be identified as *differential operator* $du^i \partial_i$ act on a set of functions $\mathbf{p}$. In abbreviated notation, we have

$$\boxed{d\mathbf{p} = du^i \partial_i = du^i \otimes \partial_i := \vartheta \quad (\mathbf{p}_i \longrightarrow \partial_i) \, ,} \tag{83}$$

where we use ∂_i to abbreviate the basis vector $\mathbf{p}_i = \partial_i \mathbf{p}$, *i.e.*, the vector ∂_i should be regarded as a *differential operator* act on some functions. Here we call $\vartheta = d\mathbf{p}$ the *canonical 1-form* or *soldering form*, which is a *vector-valued 1-form* (1-form carries a vector). For any vector $\boldsymbol{V} = V^k \partial_k$ on $\mathcal{M}$, we apply ϑ on $\boldsymbol{V}$ and obtain

$$\vartheta(\boldsymbol{V}) = V^k du^i (\partial_k) \partial_i = V^k \delta^i_k \partial_i = V^i \partial_i = \boldsymbol{V} \, . \tag{84}$$

It is apparent that ϑ is an *identity map* for a vector.

Now we will define the general inner product for two basis vectors ∂_i and ∂_j instead of dot product as

$$\boxed{g(\partial_i, \partial_j) := g_{ij}\,.} \tag{85}$$

Therefore, for any two vectors $\boldsymbol{V} = V^i\partial_i$ and $\boldsymbol{W} = W^j\partial_j$ on $\mathcal{M}$, we have

$$g(\boldsymbol{V}, \boldsymbol{W}) = g(V^i\partial_i, W^j\partial_j) = V^iW^j g(\partial_i, \partial_j) = V^iW^j g_{ij} = V_jW^j = V^iW_i\,. \tag{86}$$

In general we also have

$$\bar{g}(\partial_a, \partial_b) := \bar{g}_{ab}\,, \tag{87}$$

where

$$\partial_a := \frac{\partial}{\partial x^a} \quad (a, b = 1, 2, 3)\,. \tag{88}$$

We call $\{x^a\}$ the *Gauss normal coordinates* or *synchronous coordinates* if $\bar{g}_{i3} = 0$ and $\bar{g}_{33} = 1$, *i.e.*, ∂_3 is an unit normal vector of $\mathcal{M}$, which is proportional to **n**.

Furthermore, we can define a *metric tensor*

$$\boxed{\bar{g} = \bar{g}_{ab}dx^adx^b = \delta_{ab}dx^adx^b = ds^2} \tag{89}$$

as a line inteval of $\mathbb{E}^3$, and it is clear that $\bar{g}_{i3}$ is one of the component of $\bar{g}$. If we assume that $x^1 = x = x(u, v)$, $x^2 = y = y(u, v)$ and $x^3 = z = z(u, v)$ on $\mathcal{M}$. We have basis vectors $\frac{\partial}{\partial u^i}$ spanned by $\frac{\partial}{\partial x^a}$ as

$$\begin{cases} \dfrac{\partial}{\partial u} = \dfrac{\partial x}{\partial u}\dfrac{\partial}{\partial x} + \dfrac{\partial y}{\partial u}\dfrac{\partial}{\partial y} + \dfrac{\partial z}{\partial u}\dfrac{\partial}{\partial z} = \dfrac{\partial x^a}{\partial u}\dfrac{\partial}{\partial x^a}\,, \\ \dfrac{\partial}{\partial v} = \dfrac{\partial x}{\partial v}\dfrac{\partial}{\partial x} + \dfrac{\partial y}{\partial v}\dfrac{\partial}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial}{\partial z} = \dfrac{\partial x^a}{\partial v}\dfrac{\partial}{\partial x^a}\,, \end{cases} \implies \quad \frac{\partial}{\partial u^i} = \frac{\partial x^a}{\partial u^i}\frac{\partial}{\partial x^a} := h^a{}_i\frac{\partial}{\partial x^a}\,, \tag{90}$$

where $h^a{}_i$ is a *projection operator* of the vector in $\mathbb{E}^3$ and $i, j = 1, 2$. The component of metric tensor g of $\mathcal{M}$ can be given by

$$g_{ij} = g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) \equiv \bar{g}\left(h^a{}_i\frac{\partial}{\partial x^a}, h^b{}_j\frac{\partial}{\partial x^b}\right) = h^a{}_ih^b{}_j\,\bar{g}\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) = h^a{}_ih^b{}_j\,\bar{g}_{ab}\,, \tag{91}$$

which is the projection of $\bar{g}_{ab}$ of $\mathbb{E}^3$ onto $\mathcal{M}$. We can define an projection operation **P** of differential dx^a in $\mathbb{E}^3$ onto $\mathcal{M}$ which is called the *pullback* (a map for *contravariant* quantities) of 1-form dx^a:

$$\mathbf{P}(dx^a) = \frac{\partial x^a}{\partial u^i}du^i = h^a{}_idu^i\,, \tag{92}$$

i.e., $\mathbf{P}(dx^a)$ can be spanned by du^i on $\mathcal{M}$. As a consequence, a line inteval $ds^2\big|_{\mathcal{M}} = \mathbf{P}(\bar{g})$ on $\mathcal{M}$ is obtained by

$$\mathbf{P}(\bar{g}) = \mathbf{P}(\bar{g}_{ab}dx^adx^b) = \bar{g}_{ab}\big(h^a{}_ih^b{}_jdu^idu^j\big) = \big(\bar{g}_{ab}h^a{}_ih^b{}_j\big)du^idu^j := g_{ij}du^idu^j = g = \mathbf{I}\,. \tag{93}$$

Therefore, the first fundamental form $\mathbf{I} = g = ds^2\big|_{\mathcal{M}}$ of $\mathcal{M}$ can be regarded as a projection of metric tensor $\bar{g}$ with

$$\boxed{g_{ij} := \bar{g}_{ab}h^a{}_ih^b{}_j = \bar{g}_{ab}\frac{\partial x^a}{\partial u^i}\frac{\partial x^b}{\partial u^j}\,.} \tag{94}$$

We called that g is a *induced metric* obtained by the pullback of $\bar{g}$.

Supplement (Interior product)**.** We define an anti-derivation on exterior differential p-forms ω for a vector $\boldsymbol{X}$ called *interior product* with respect to $\boldsymbol{X}$. It sends an exterior p-form to an exterior $(p-1)$-form. We consider an 1-forms ω, the interior product of ω with respect to a vector $\boldsymbol{X}$ is

$$\boxed{\begin{aligned} \iota_{\boldsymbol{X}}\omega \overset{\text{or}}{=} \boldsymbol{X}\rfloor\omega &:= \omega(\boldsymbol{X}) && (95a)\\ &= X^i\omega_j du^j(\partial_i) = X^i\omega_j\delta_i^j = X^j\omega_j\,. && (95b)\end{aligned}}$$

Therefore, we have

$$\iota_{\partial_i}(du^j) = du^j(\partial_i) = \delta_i^j\,. \tag{96}$$

We note that:

- For 0-form f (a scalar), the interior product is vanished $\iota_{\boldsymbol{X}}f = 0$ because of no (-1)-form.
- The second action of $\iota_{\boldsymbol{X}}^2 = 0$. It can be shown that by considering an exterior 3-form $\omega = (1/3!)\omega_{ijk}du^i\wedge du^j\wedge du^k$, we have vanished second interior product by $\boldsymbol{X}$

$$\begin{aligned}\iota_{\boldsymbol{X}}^2\omega &= \iota_{\boldsymbol{X}}\iota_{\boldsymbol{X}}\left(\frac{1}{3!}\omega_{ijk}du^i\wedge du^j\wedge du^k\right)\\ &= \iota_{\boldsymbol{X}}\left(\frac{1}{3!}X^k\omega_{ijk}\Big(du^i(\partial_l)du^j\wedge du^k\right.\\ &\qquad\left.+(-1)^1du^i\wedge du^j(\partial_l)du^k+(-1)^2du^i\wedge du^j du^k(\partial_l)\Big)\right)\\ &= \iota_{\boldsymbol{X}}\left(\frac{1}{3!}X^l\Big(\omega_{ljk}du^j\wedge du^k - \underbrace{\omega_{ilk}}_{-\omega_{lik}}du^i\wedge du^k + \underbrace{\omega_{ijl}}_{+\omega_{lij}}du^i\wedge du^j\Big)\right)\\ &= \iota_{\boldsymbol{X}}\left(\frac{1}{2}X^l\omega_{ljk}du^j\wedge du^k\right)\\ &= X^mX^l\left(\frac{1}{2}\omega_{ljk}\Big(du^j(\partial_m)du^k+(-1)^1du^j du^k(\partial_m)\Big)\right)\\ &= X^mX^l\left(\frac{1}{2}\omega_{lmk}du^k - \underbrace{\omega_{ljm}}_{-\omega_{lmj}}du^j\right) = \overbrace{X^mX^l}^{\text{symmetric in } l,m}\underbrace{\omega_{lmk}}_{\text{anti-symmetric in } l,m}du^k = 0\,.\end{aligned} \tag{97}$$

  However, $\iota_{\boldsymbol{Y}}\iota_{\boldsymbol{X}} \neq 0$, *e.g.* the interior product of an exterior 3-form ω by $\boldsymbol{X}$ and $\boldsymbol{Y}$ should be an 1-form $Y^mX^l\omega_{lmk}du^k \neq 0$.

Supplement (Isomorphism between tangent and cotangent space)**.** The 1-form du^i is defined on the cotangent space which is dual to the basis $\frac{\partial}{\partial u^i} \equiv \partial_i$ on the tangent space. We can define a *linear* map $\psi : T\mathcal{M} \longrightarrow T^*\mathcal{M}$. For vectors $\boldsymbol{X}, \boldsymbol{Y} \in T\mathcal{M}$ and $\alpha \in T^*\mathcal{M}$ the 1-form corresponding to vector $\boldsymbol{X}$, then we define

$$\boxed{\langle\alpha, \boldsymbol{Y}\rangle := \alpha(\boldsymbol{Y}) = \iota_{\boldsymbol{Y}}\alpha = g(\boldsymbol{X}, \boldsymbol{Y})\,,} \tag{98}$$

where the $\langle\bullet,\bullet\rangle$ with two slots is a kind of inner product defined between the tangent and cotangent space and

$$\boxed{\alpha := \psi(\boldsymbol{X})\,.} \tag{99}$$

Therefore, we can also write the corresponding 1-form α as

$$\boxed{\alpha(\bullet) = g(\boldsymbol{X}, \bullet)\,,} \tag{100}$$

which can be recognized by

$$\begin{aligned}\alpha &:= \frac{1}{2} g_{ij}\big(du^i(\boldsymbol{X})du^j + du^i du^j(\boldsymbol{X})\big)\\ &= \frac{1}{2} g_{ij} X^k\big(du^i(\partial_k)du^j + du^i du^j(\partial_k)\big)\\ &= \frac{1}{2} g_{ij} X^k\big(\delta^i_k du^j + du^i\delta^j_k\big)\\ &= X_i du^i\,.\end{aligned} \tag{101}$$

By choosing vector $\boldsymbol{X} = \partial_i$, we have an 1-form $\psi(\partial_i) = \psi_{ij} du^j$, which leads to

$$g_{ij} = g(\partial_i, \partial_j) = \langle\psi(\partial_i), \partial_j\rangle = \psi_{ik}\langle du^k, \partial_j\rangle = \psi_{ik}\delta^k_j = \psi_{ij}\,. \tag{102}$$

Therefore, we have

$$\boxed{\psi(\partial_i) = g_{ij} du^j := du_i} \tag{103}$$

called the *reciprocal basis* of du^i in $T^*\mathcal{M}$. It is apparent that g_{ij} transforms du^i to its *reciprocal basis* du_j.

In addition, we have a *linear* inverse map $\psi^{-1} : T^*\mathcal{M} \longrightarrow T\mathcal{M}$ such that

$$\boxed{\boldsymbol{X} = \psi^{-1}(\alpha) := \varphi(\alpha)\,.} \tag{104}$$

For $\alpha = du^i$, the inverse map of du^i can be written as $\psi^{-1}(du^i) = \varphi^{ij}\partial_j$. If we take $\alpha = du^i$ and $Y = \partial_j$ in (98), then

$$\delta^i_j = du^i(\partial_j) = g\big(\varphi^{ik}\partial_k, \partial_j\big) = \varphi^{ik} g\big(\partial_k, \partial_j\big) = \varphi^{ik} g_{kj}\,, \tag{105}$$

therefore, $\varphi^{ik} = g^{ik}$ and

$$\boxed{\psi^{-1}(du^i) = g^{ij}\partial_j := \partial^i} \tag{106}$$

is the reciprocal basis of ∂_i. We assume that $\beta = \psi(\boldsymbol{Y})$ and define the inner product in $T^*\mathcal{M}$ which is also denoted by g

$$\boxed{g(\alpha, \beta) := g\big(\psi^{-1}(\alpha), \psi^{-1}(\beta)\big)\,.} \tag{107}$$

The definition leads to the following relation by choosing $\alpha = du^i$ and $\beta = du^j$

$$g(du^i, du^j) = g\big(\psi^{-1}(du^i), \psi^{-1}(du^j)\big) = g\big(\partial^k, \partial^j\big) = g^{ik} g^{jl} g\big(\partial_k, \partial_l\big) = g^{ij}\,. \tag{108}$$

Now we can clearly express a vector $\boldsymbol{X} = X^i\partial_i$ as an inverse map ψ^{-1} of an 1-form α with the help of (106):

$$\boldsymbol{X} = X^i\partial_i = X^i g_{ij}\partial^j = \underbrace{X_j\partial^j}_{\text{a vector!}} = X_j\psi^{-1}(du^j) = \psi^{-1}(\underbrace{X_j du^j}_{\text{an 1-form!}}) = \psi^{-1}(\alpha)\,. \tag{109}$$

As a result, we conclude that the metric tensor g (not component g_{ij} or g^{ij}) turns a vector (1-form) into a 1-form (vector). The component of metric tensor g^{ij} (g_{ij}) transforms the a vector ∂_i (1-form du^i) to its corresponding reciprocal basis ∂^i (du_i).

Normal vector of the surface We would like to look for an orthonormal frame $(\mathbf{p}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathcal{M}$. Under the *Gram-Schmit* procedure, we can define

$$\boxed{\mathbf{e}_1 := \frac{\mathbf{p}_u}{|\mathbf{p}_u|}} \tag{110}$$

and

$$\boxed{\mathbf{e}_2 := \frac{\mathbf{p}_v - (\mathbf{p}_v \cdot \mathbf{e}_1)\mathbf{e}_1}{|\mathbf{p}_v - (\mathbf{p}_v \cdot \mathbf{e}_1)\mathbf{e}_1|}\,.} \tag{111}$$

Therefore, we have

$$\boxed{\mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 := \mathbf{n}\,,} \tag{112}$$

which is an *unit normal vector* of $\mathcal{M}$.

The unit normal vector $\mathbf{n} = \mathbf{n}(u, v)$ can also be obtained by

$$\boxed{\mathbf{n}(u, v) = \frac{\mathbf{p}_u \wedge \mathbf{p}_v}{|\mathbf{p}_u \wedge \mathbf{p}_v|}\,.} \tag{113}$$

The corresponding differential $d\mathbf{n}$ is

$$d\mathbf{n} = \frac{\partial \mathbf{n}}{\partial u}du + \frac{\partial \mathbf{n}}{\partial v}dv = \mathbf{n}_u du + \mathbf{n}_v dv\,. \tag{114}$$

However, we have:
(i) $\mathbf{n} \perp \mathbf{p}_u \implies \mathbf{n} \cdot \mathbf{p}_u = 0$, which have the equations of the partial derivative with respect to u and v are

$$\begin{cases} \partial_u : \mathbf{n}_u \cdot \mathbf{p}_u + \mathbf{n} \cdot \mathbf{p}_{uu} = 0 \implies \mathbf{n} \cdot \mathbf{p}_{uu} = -\mathbf{n}_u \cdot \mathbf{p}_u := L\,, & (115\text{a}) \\ \partial_v : \mathbf{n}_v \cdot \mathbf{p}_u + \mathbf{n} \cdot \mathbf{p}_{uv} = 0 \implies \mathbf{n} \cdot \mathbf{p}_{uv} = -\mathbf{n}_v \cdot \mathbf{p}_u := M\,. & (115\text{b}) \end{cases}$$

(ii) $\mathbf{n} \perp \mathbf{p}_v \implies \mathbf{n} \cdot \mathbf{p}_v = 0$, we obtain

$$\begin{cases} \partial_u : \mathbf{n}_u \cdot \mathbf{p}_v + \mathbf{n} \cdot \mathbf{p}_{vu} = 0 \implies \mathbf{n} \cdot \mathbf{p}_{vu} = -\mathbf{n}_u \cdot \mathbf{p}_v := M\,, & (116\text{a}) \\ \partial_v : \mathbf{n}_v \cdot \mathbf{p}_v + \mathbf{n} \cdot \mathbf{p}_{vv} = 0 \implies \mathbf{n} \cdot \mathbf{p}_{vv} = -\mathbf{n}_v \cdot \mathbf{p}_v := N\,. & (116\text{b}) \end{cases}$$

Second fundamental (quadratic) form According to (62) and (114), we can define a quadratic form

$$\begin{aligned} \mathbf{II} &:= -d\mathbf{p} \cdot d\mathbf{n} && (117\text{a}) \\ &= -(\mathbf{p}_u du + \mathbf{p}_v dv)(\mathbf{n}_u du + \mathbf{n}_v dv) \\ &= -(\mathbf{p}_u \cdot \mathbf{n}_u\, dudu + \mathbf{p}_u \cdot \mathbf{n}_v\, dudv + \mathbf{p}_v \cdot \mathbf{n}_u\, dudv + \mathbf{p}_v \cdot \mathbf{n}_v\, dvdv) \\ &= \underbrace{\mathbf{n} \cdot \mathbf{p}_{uu}}_{L} dudu + \underbrace{\mathbf{n} \cdot \mathbf{p}_{uv}}_{M} dudv + \underbrace{\mathbf{n} \cdot \mathbf{p}_{vu}}_{M} dudv + \underbrace{\mathbf{n} \cdot \mathbf{p}_{vv}}_{N} dvdv \\ &= L\, dudu + 2M\, dudv + N\, dudv && (117\text{b}) \\ &= \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} && (117\text{c}) \end{aligned}$$

called the *second fundamental form* of $\mathcal{M}$. We define the second fundamental form as a tensor given by

$$\boxed{\mathbf{II} = b_{ij} du^i du^j} \tag{118}$$

with the component of matrix form as

$$b_{ij} = \mathbf{n} \cdot \mathbf{p}_{ij} = -\mathbf{p}_i \cdot \mathbf{n}_j \longrightarrow \begin{pmatrix} L & M \\ M & N \end{pmatrix} . \tag{119}$$

Remark. The second fundamental form describes the *shape* of $\mathcal{M}$ and how the surface $\mathcal{M}$ embedded in $\mathbb{E}^3$. It is an *extrinsic* property of $\mathcal{M}$ and we call the component b_{ij} the *extrinsic curvature*.

Now we would like to discuss decomposition formulas of the derivative vector of frame $(\mathbf{p}; \mathbf{p}_u, \mathbf{p}_v, \mathbf{n})$. We follow the principle:

- *Any vector in the space can be spanned by the basis $\mathbf{p}_u$, $\mathbf{p}_v$ and $\mathbf{n}$.*

Gauss formulas We take the partial derivative of $\mathbf{p}_u$ and $\mathbf{p}_v$ with respect to u and v:

$$\begin{cases} \mathbf{p}_{uu} := \dfrac{\partial}{\partial u}\mathbf{p}_u = (\Gamma_u)^u{}_u\, \mathbf{p}_u + (\Gamma_u)^v{}_u\, \mathbf{p}_v + (\Gamma_u)^{\mathbf{n}}{}_u\, \mathbf{n} \\ \qquad\quad = (\Gamma_u)^u{}_u\, \mathbf{p}_u + (\Gamma_u)^v{}_u\, \mathbf{p}_v + \underbrace{(\mathbf{p}_{uu} \cdot \mathbf{n})}_{L}\mathbf{n} , & (120a) \\ \mathbf{p}_{uv} := \dfrac{\partial}{\partial v}\mathbf{p}_u = (\Gamma_u)^u{}_v\, \mathbf{p}_u + (\Gamma_u)^v{}_v\, \mathbf{p}_v + (\Gamma_u)^{\mathbf{n}}{}_v\, \mathbf{n} \\ \qquad\quad = (\Gamma_u)^u{}_v\, \mathbf{p}_u + (\Gamma_u)^v{}_v\, \mathbf{p}_v + \underbrace{(\mathbf{p}_{uv} \cdot \mathbf{n})}_{M}\mathbf{n} , & (120b) \\ \mathbf{p}_{vu} := \dfrac{\partial}{\partial u}\mathbf{p}_v = (\Gamma_v)^u{}_u\, \mathbf{p}_u + (\Gamma_v)^v{}_u\, \mathbf{p}_v + (\Gamma_v)^{\mathbf{n}}{}_u\, \mathbf{n} \\ \qquad\quad = (\Gamma_v)^u{}_u\, \mathbf{p}_u + (\Gamma_v)^v{}_u\, \mathbf{p}_v + \underbrace{(\mathbf{p}_{vu} \cdot \mathbf{n})}_{M}\mathbf{n} , & (120c) \\ \mathbf{p}_{vv} := \dfrac{\partial}{\partial v}\mathbf{p}_v = (\Gamma_v)^u{}_v\, \mathbf{p}_u + (\Gamma_v)^v{}_v\, \mathbf{p}_v + (\Gamma_v)^{\mathbf{n}}{}_v\, \mathbf{n} \\ \qquad\quad = (\Gamma_v)^u{}_v\, \mathbf{p}_u + (\Gamma_v)^v{}_v\, \mathbf{p}_v + \underbrace{(\mathbf{p}_{vv} \cdot \mathbf{n})}_{N}\mathbf{n} . & (120d) \end{cases}$$

We call these set of equations the *Gauss formulas*. We identify the coefficients

$$(\Gamma_a)^c{}_b \equiv \Gamma^c{}_{ab} , \tag{121}$$

e.g., $(\Gamma_{\mathbf{n}})^u{}_v = \Gamma^1{}_{n2}$, then Gauss formulas (120) can be written as

$$\mathbf{p}_{ij} = \Gamma^k{}_{ij}\mathbf{p}_k + \Gamma^{\mathbf{n}}{}_{ij}\mathbf{n} = \Gamma^k{}_{ij}\mathbf{p}_k + b_{ij}\mathbf{n} \quad (i, j, k = 1, 2) , \tag{122}$$

where the coefficients are obtained by

$$\Gamma_{kij} = \Gamma^l{}_{ij} g_{lk} = \underbrace{\Gamma^l{}_{ij}\mathbf{p}_l}_{\mathbf{p}_{ij} - b_{ij}\mathbf{n}} \cdot \mathbf{p}_k = \mathbf{p}_{ij} \cdot \mathbf{p}_k \tag{123}$$

and

$$b_{ij} = \Gamma^{\mathbf{n}}{}_{ij} = \mathbf{n} \cdot \mathbf{p}_{ij} . \tag{124}$$

We note that Γ_{kij} and b_{ij} are *symmetric* in i, j.

Remark. The vectors $d\mathbf{p}$ and $d\mathbf{p}_i$ in terms of the differential form are given by

$$\boxed{d\mathbf{p} = (du^i\partial_i)\mathbf{p} = du^i\,\mathbf{p}_i} \tag{125}$$

and

$$\boxed{d\mathbf{p}_i = d(\partial_i\mathbf{p}) = du^j(\partial_j\partial_i\mathbf{p}) = du^j(\Gamma^k{}_{ij}\mathbf{p}_k + b_{ij}\mathbf{n}) := \Gamma^k{}_i\mathbf{p}_k + b_i\mathbf{n} = \Gamma^k{}_i\mathbf{p}_k + \Gamma^n{}_i\mathbf{n}} \tag{126}$$

respectively, where $\Gamma^k{}_i := \Gamma^k{}_{ij}du^j$ is called *connection form* and $b_i := b_{ij}du^j = \Gamma^{\mathbf{n}}{}_{ij}du^j = \Gamma^{\mathbf{n}}{}_i$.

Weingarten formulas Due to $\mathbf{n}\cdot\mathbf{n} = 1$, we have

$$\begin{cases} \partial_u : \mathbf{n}_u \cdot \mathbf{n} = 0 \quad \Longrightarrow \quad \mathbf{n}_u \perp \mathbf{n}\,, & (127a)\\ \partial_v : \mathbf{n}_v \cdot \mathbf{n} = 0 \quad \Longrightarrow \quad \mathbf{n}_v \perp \mathbf{n}\,. & (127b)\end{cases}$$

Therefore, $\mathbf{n}_u$ and $\mathbf{n}_v$ do not contain the component of $\mathbf{n}$. We can assume that

$$\begin{cases} \mathbf{n}_u = A\,\mathbf{p}_u + B\,\mathbf{p}_v = (\Gamma_{\mathbf{n}})^u_u\,\mathbf{p}_u + (\Gamma_{\mathbf{n}})^v_u\,\mathbf{p}_v\,, & (128a)\\ \mathbf{n}_v = C\,\mathbf{p}_u + D\,\mathbf{p}_v = (\Gamma_{\mathbf{n}})^u_v\,\mathbf{p}_u + (\Gamma_{\mathbf{n}})^v_v\,\mathbf{p}_v\,, & (128b)\end{cases}$$

which is called the *Weingarten formulas*. We calculate the inner product of $\mathbf{n}_u\cdot\mathbf{p}_u$ and $\mathbf{n}_u\cdot\mathbf{p}_v$:

$$\begin{cases} \mathbf{n}_u\cdot\mathbf{p}_u = -L = EA + FB\\ \mathbf{n}_u\cdot\mathbf{p}_v = -M = FA + GB \end{cases} \Longrightarrow \quad \boxed{A = \frac{FM - GL}{EG - F^2}} \quad \text{and} \quad \boxed{B = \frac{FL - EM}{EG - F^2}\,.} \tag{129}$$

Similarly, we have

$$\boxed{C = \frac{FN - GM}{EG - F^2}} \quad \text{and} \quad \boxed{D = \frac{FM - EN}{EG - F^2}\,.} \tag{130}$$

The Weingarten formulas (128) can be written by

$$\boxed{\mathbf{n}_j = \Gamma^k{}_{\mathbf{n}j}\mathbf{p}_k\,,} \tag{131}$$

where the coefficients can be calculated by

$$-b_{ij} = \mathbf{p}_i\cdot\mathbf{n}_j = \mathbf{p}_i\cdot\left(\Gamma^l{}_{\mathbf{n}j}\mathbf{p}_l\right) = g_{il}\Gamma^l{}_{\mathbf{n}j} = \Gamma_{i\mathbf{n}j} \quad \Longrightarrow \quad \boxed{\Gamma^k{}_{\mathbf{n}j} = g^{ki}\Gamma_{i\mathbf{n}j} = -g^{ki}b_{ij} := -b^k{}_j\,.} \tag{132}$$

As a result, we obtain

$$\boxed{\mathbf{n}_j = -b^k{}_j\mathbf{p}_k\,.} \tag{133}$$

Remark. The Weingarten formula written in the differential form is given by

$$\boxed{d\mathbf{n} = (-b^k{}_j du^j)\mathbf{p}_k = -b^k\mathbf{p}_k := \Gamma^k{}_{\mathbf{n}}\mathbf{p}_k} \tag{134}$$

Acceleration (curvature) vector We have an acceleration (curvature) vector $\mathbf{p}''(s)$ parametrized by s, which can be decomposed by tangential and normal parts

$$\boxed{\mathbf{p}'' = \mathbf{p}_t'' + \mathbf{p}_n'' := \boldsymbol{\kappa}_g + \boldsymbol{\kappa}_n\,,} \tag{135}$$

where the tangential part $\boldsymbol{\kappa}_g = \kappa_g \mathbf{t}$ and normal part $\boldsymbol{\kappa}_n = \kappa_n \mathbf{n}$ are called *geodesic curvature* and *normal curvature* respectively.

Remark. We can identify $\mathbf{p}'' := \boldsymbol{a}$ the acceleration vector, therefore, (135) can be read as $\boldsymbol{a} = \boldsymbol{a}_{\mathrm{T}} + \boldsymbol{a}_{\mathrm{N}}$ with the tangent acceleration vector $\boldsymbol{a}_{\mathrm{T}} := \mathbf{p}_t''$ and normal acceleration vector $\boldsymbol{a}_{\mathrm{N}} := \mathbf{p}_n''$.

According to (11), we have $\mathbf{p}'' \cdot \mathbf{p}' = 0$. We would like to discuss the geodesic curvature, we take the inner product of $\mathbf{p}_t''$ with $\mathbf{n}$ and $\mathbf{p}'$ respectively:

$$\begin{cases} \mathbf{p}_t'' \cdot \mathbf{n} := 0\,, \\ \mathbf{p}_t'' \cdot \mathbf{p}' := \left(\mathbf{p}_t'' + \mathbf{p}_n''\right) \cdot \mathbf{p}' = \mathbf{p}'' \cdot \mathbf{p}' = 0\,, \end{cases} \quad \Longrightarrow \quad \mathbf{p}_t'' \propto \mathbf{t} := \mathbf{n} \wedge \mathbf{p}'\,, \tag{136}$$

then we can have $\mathbf{p}_t'' = \kappa_g \mathbf{t} = \kappa_g(\mathbf{n} \wedge \mathbf{p}')$.

Normal curvature of a curve We would like to discuss normal curvature first, and define $\boldsymbol{\kappa}_n = \kappa_n \mathbf{n}$, so

$$\kappa_n = \boldsymbol{\kappa}_n \cdot \mathbf{n} = (\mathbf{p}'' - \boldsymbol{\kappa}_g) \cdot \mathbf{n} = \mathbf{p}'' \cdot \mathbf{n} - \underbrace{\boldsymbol{\kappa}_g \cdot \mathbf{n}}_{0}\,. \tag{137}$$

However we also have

$$\mathbf{p}' \cdot \mathbf{n} = 0 \quad \Longrightarrow \quad \mathbf{p}'' \cdot \mathbf{n} + \mathbf{p}' \cdot \mathbf{n}' = 0\,. \tag{138}$$

Therefore, κ_n can be calculated by

$$\begin{aligned} \kappa_n &= -\mathbf{p}' \cdot \mathbf{n}' \\ &= -(\mathbf{p}_u u' + \mathbf{p}_v v') \cdot (\mathbf{n}_u u' + \mathbf{n}_v v') \\ &= L\, u'u' + 2M\, u'v' + N\, v'v'\,. \end{aligned} \tag{139}$$

However $\mathbf{p}' = d\mathbf{p}/ds$ and $\mathbf{n}' = d\mathbf{n}/ds$, which leads to

$$\boxed{\kappa_n = -\frac{d\mathbf{p}}{ds} \cdot \frac{d\mathbf{n}}{ds} = \frac{-d\mathbf{p} \cdot d\mathbf{n}}{ds^2} = \frac{\mathbf{II}}{\mathbf{I}}\,.} \tag{140}$$

Remark. We have the norm of tangent vector

$$1 = |\mathbf{p}'| = Eu'u' + 2Fu'v' + Gv'v' = \frac{\mathbf{I}}{ds^2}\,. \tag{141}$$

According to (83)

$$\mathbf{p}' = \frac{d\mathbf{p}}{ds} = \frac{du^i}{ds}\partial_i = u^{i\prime}\partial_i\,, \tag{142}$$

we have

$$
\begin{aligned}
\mathbf{I}(\mathbf{p}',\mathbf{p}') &= g_{kl}du^k du^l(\mathbf{p}',\mathbf{p}') \\
&= g_{kl}u^{i\prime}u^{j\prime}du^k(\partial_i)\,du^l(\partial_j) \\
&= g_{kl}u^{i\prime}u^{j\prime}\delta_i^k\delta_j^l \\
&= g_{ij}u^{i\prime}u^{j\prime} && (143a) \\
&= Eu'u' + 2Fu'v' + Gv'v' = 1\,. && (143b)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbf{II}(\mathbf{p}',\mathbf{p}') &= b_{ij}u^{i\prime}u^{j\prime} && (144a) \\
&= L\,u'u' + 2M\,u'v' + N\,v'v' = \kappa_n\,. && (144b)
\end{aligned}
$$

We finally obtain

$$
\kappa_n = \frac{\mathbf{II}(\mathbf{p}',\mathbf{p}')}{\mathbf{I}(\mathbf{p}',\mathbf{p}')} = \mathbf{II}(\mathbf{p}',\mathbf{p}')\,. \tag{145}
$$

We assume that κ_n has value of λ, which gives the relation

$$
\mathbf{II} = \lambda \mathbf{I}\,. \tag{146}
$$

One can divide (146) by ds^2 and then obtain

$$
\frac{\mathbf{II}}{ds^2} = \lambda\frac{\mathbf{I}}{ds^2} \quad\Longrightarrow\quad L\,u'u' + 2M\,u'v' + N\,v'v' = \lambda\big(Eu'u' + 2Fu'v' + Gv'v'\big)\,, \tag{147}
$$

where λ can be recognized as the Lagrangian multiplier with constraint $Eu'u' + 2Fu'v' + Gv'v' = 1$. By looking for the extrema λ of $\kappa_n = \mathbf{II}/ds^2$, we take the partial derivative of (147) with respect to $u^{i\prime}$, which leads to the equation of matrix form

$$
\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = \lambda\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} \quad\Longrightarrow\quad \begin{pmatrix} L-\lambda E & M-\lambda F \\ M-\lambda F & N-\lambda G \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = 0\,, \tag{148}
$$

which means

$$
\boxed{(b_{ij} - \lambda g_{ij})u^{j\prime} = 0\,.} \tag{149}
$$

We have to look for the *non-trivial* solutions, *i.e.*,

$$
\begin{aligned}
& \det(b_{ij} - \lambda g_{ij}) = 0\,, \\
\Longrightarrow\quad & (EG - F^2)\lambda^2 - \big(EN + GL - 2FM\big)\lambda + LN - M^2 = 0\,, \\
\Longrightarrow\quad & \boldsymbol{g}\lambda^2 - \big(EN + GL - 2FM\big)\lambda + \boldsymbol{b} = 0\,, && (150)
\end{aligned}
$$

where we define

$$
\begin{cases}
\boldsymbol{g} := \det(g_{ij}) = EG - F^2\,, & (151a) \\
\boldsymbol{b} := \det(b_{ij}) = LN - M^2\,. & (151b)
\end{cases}
$$

As a result, we have the sum and product of two solutions λ_1 and λ_2

$$
\lambda_1 + \lambda_2 = \frac{EN + GL - 2FM}{\boldsymbol{g}} \quad\text{and}\quad \lambda_1\lambda_2 = \frac{\boldsymbol{b}}{\boldsymbol{g}}\,. \tag{152}
$$

Gauss curvature We define the *Gauss curvature* (or called *total curvature*) as product of two curvatures:

$$\boxed{K := \lambda_1\lambda_2 = \frac{\boldsymbol{b}}{\boldsymbol{g}}\,.} \tag{153}$$

Mean curvature The *mean curvature* is defined by the mean value of sum of two curvatures:

$$\boxed{H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{EN + GL - 2FM}{2\,\boldsymbol{g}}\,.} \tag{154}$$

Remark. The value λ_α ($\alpha = 1, 2$) is called the *principal curvature* of κ_n. By substituting λ_α into the equation (149), the corresponding solution of vector $\mathbf{p}'_{(\alpha)} = u^{j\,\prime}_{(\alpha)}\mathbf{p}_j$ or $d\mathbf{p}_{(\alpha)} = du^j_{(\alpha)}\mathbf{p}_j$ is called the *principal direction*.

Example (Cylindrical surface in $\mathbb{E}^3$). A surface parallel with z-axis can be described by

$$\mathbf{p} = \big(x, y, z\big) = \big(x(u), y(u), v\big) \quad \Longrightarrow \quad d\mathbf{p} = \big(dx, dy, dz\big) = \big(x'du, y'du, dv\big)\,. \tag{155}$$

A cylindrical surface need to have a constraint with

$$\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 = x'^{\,2} + y'^{\,2} = 1\,. \tag{156}$$

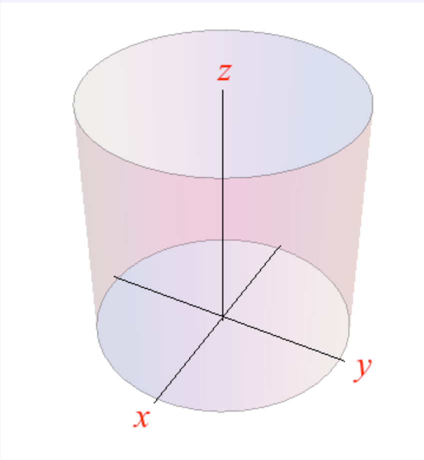


Figure 7: A cylinder.

A cylinder is parametrized by

$$\mathbf{p}(u, v) = \big(x(u), y(u), v\big) \quad \Longrightarrow \quad d\mathbf{p} = \mathbf{p}_u du + \mathbf{p}_v dv = \big(x'du, y'du, dv\big)\,, \tag{157}$$

where

$$\begin{cases} \mathbf{p}_u = \big(x'du, y'du, 0\big)\,, \\ \mathbf{p}_v = \big(0, 0, 1\big)\,. \end{cases} \quad \Longrightarrow \quad \mathbf{p}_u \wedge \mathbf{p}_v = \big(y', -x', 0\big)\,. \tag{158}$$

and the normal vector is

$$\mathbf{n} = \frac{\mathbf{p}_u \wedge \mathbf{p}_v}{|\mathbf{p}_u \wedge \mathbf{p}_v|} = \frac{1}{y'^2 + x'^2}\big(y', -x', 0\big) = \big(y', -x', 0\big) \tag{159}$$

and

$$d\mathbf{n} = \mathbf{n}_u du + \mathbf{n}_v dv = \big(y'' du, -x'' du, 0\big)\,. \tag{160}$$

with

$$\begin{cases} \mathbf{n}_u = \big(y'', -x'', 0\big)\,, \\ \mathbf{n}_v = \big(0, 0, 0\big)\,. \end{cases} \tag{161}$$

Then we have first fundamental form

$$\mathbf{I} = d\mathbf{p} \cdot d\mathbf{p} = \big(x'^2 + y'^2\big) du^2 + dv^2 = du^2 + dv^2\,, \tag{162}$$

where

$$E = G = 1, \quad F = 0\,. \tag{163}$$

The second fundamental form can be obtained by

$$\begin{aligned} \mathbf{II} &= -d\mathbf{p} \cdot d\mathbf{n} \\ &= -(\mathbf{p}_u \cdot \mathbf{n}_u\, du du + \underbrace{\mathbf{p}_u \cdot \mathbf{n}_v}_{0}\, du dv + \underbrace{\mathbf{p}_v \cdot \mathbf{n}_u}_{0}\, du dv + \underbrace{\mathbf{p}_v \cdot \mathbf{n}_v}_{0}\, dv dv) \\ &= -\big(x' y'' - y' x''\big) du^2 \\ &= \big(y' x'' - x' y''\big) du^2\,, \end{aligned} \tag{164}$$

where

$$L = y' x'' - x' y'', \quad M = N = 0\,. \tag{165}$$

So we have

$$\boldsymbol{g} = 1\,, \quad \boldsymbol{b} = 0\,, \quad EN + GL - 2FM = y' x'' - x' y''\,. \tag{166}$$

As a result, we obtain

$$\text{Gauss curvature:} \quad K = \lambda_1 \lambda_2 = 0, \tag{167}$$

$$\text{mean curvature:} \quad H = \frac{1}{2}\big(\lambda_1 + \lambda_2\big) = \frac{1}{2}\big(y' x'' - x' y''\big)\,. \tag{168}$$

We can solve the above equations to obtain

$$\lambda_1 = 0, \quad \text{and} \quad \lambda_2 = y' x'' - x' y''\,. \tag{169}$$

Geodesic equations The tangent vector parametrized by s is

$$\mathbf{p}'(s) = \mathbf{p}_i u^{i\,\prime}\,, \tag{170}$$

it leads to the acceleration vector is given by

$$\begin{aligned}
\mathbf{p}'' &= \mathbf{p}_i' u^{i\prime} + \mathbf{p}_i u^{i\prime\prime} \\
&= \big(\Gamma^k{}_{ij}\mathbf{p}_k + b_{ij}\mathbf{n}\big) u^{i\prime} u^{j\prime} + \mathbf{p}_i u^{i\prime\prime} \\
&= \big(u^{k\prime\prime} + \Gamma^k{}_{ij} u^{i\prime} u^{j\prime}\big)\mathbf{p}_k + b_{ij} u^{i\prime} u^{j\prime}\mathbf{n} && (171)\\
&= \mathbf{p}_t'' + \mathbf{p}_n'' && (172)\\
&= \boldsymbol{\kappa}_g + \boldsymbol{\kappa}_n\,, && (173)
\end{aligned}$$

where we have used $\mathbf{p}_i' = \mathbf{p}_{ij}(du^j/ds) = (\Gamma^k{}_{ij}\mathbf{p}_k + b_{ij}\mathbf{n})u^{j\prime}$. Now we call the curve $\mathbf{p}(s)$ *geodesic* if $\mathbf{p}'' = \mathbf{p}_n'' = \boldsymbol{\kappa}_n$, *i.e.*, the *tangential* part is vanished

$$\boxed{\boldsymbol{\kappa}_g = \mathbf{p}_t'' = \big(u^{k\prime\prime} + \Gamma^k{}_{ij} u^{i\prime} u^{j\prime}\big)\mathbf{p}_k = 0\,,} \qquad (174)$$

which means that

$$\boxed{\mathbf{p} \text{ only has the normal curvature } \boldsymbol{\kappa}_n.}$$

Because $\mathbf{p}_k$s are linear independent, we obtain

$$\boxed{u^{k\prime\prime} + \Gamma^k{}_{ij} u^{i\prime} u^{j\prime} = 0\,,} \qquad (175)$$

which is called *geodesic equations*.

Supplement (Connection and geodesic)**.** We have differential of $\mathbf{p}'$ given by

$$\begin{aligned}
d\mathbf{p}' &= d\mathbf{p}_i u^{i\prime} + \mathbf{p}_i du^i \\
&= \big(\Gamma^k{}_i \mathbf{p}_k + b_i \mathbf{n}\big) u^{i\prime} + \mathbf{p}_i du^{i\prime} \\
&= \big(du^{k\prime} + \Gamma^k{}_i u^{i\prime}\big)\mathbf{p}_k + b_i u^{i\prime}\mathbf{n}\,, && (176)
\end{aligned}$$

i.e., the symbol d is a *total* or *absolute* differential with respect to frame $(\mathbf{p}; \mathbf{p}_1, \mathbf{p}_2, \mathbf{n})$ on $\mathbb{E}^3$. The total or absolute differentiation means that we have to differentiate not only the *component* $u^{i\prime}$ but also the *basis* $\mathbf{p}_i$ of a vector $\mathbf{p}'$. We assume that $\mathbf{p}'$ can be written as

$$\mathbf{p}' = u^{i\prime}\partial_i := V^i\partial_i = \boldsymbol{V}\,. \qquad (177)$$

Then (176) can be written as

$$\boxed{d\boldsymbol{V} = \big(dV^k + \Gamma^k{}_i V^i\big)\partial_k + b_i V^i \mathbf{n}\,.} \qquad (178)$$

and the geodesic equation becomes as

$$\boxed{\big(dV^k + \Gamma^k{}_i V^i\big)\partial_k = 0\,.} \qquad (179)$$

We define the *connection* $D = du^i \otimes D_i$ on $\mathcal{M}$, which act on the function V^i and basis ∂_i are

$$\begin{cases}
DV^i = dV^i = du^j \otimes \partial_j V^i\,, & (180a)\\
D\partial_i = \Gamma^k{}_i \otimes \partial_k = du^j \otimes \Gamma^k{}_{ij}\partial_k\,, & (180b)
\end{cases}$$

respectively. We note that D act on a function as a differential d on a function. The connection act on a vector is given by

$$\boxed{D\boldsymbol{V} = \big(DV^i\big)\otimes\partial_i + V^i\big(D\partial_i\big) = dV^i\otimes\partial_i + V^i\Gamma^k{}_i\otimes\partial_k = \big(\underbrace{dV^k + \Gamma^k{}_i V^i}_{(D\boldsymbol{V})^k}\big)\otimes\partial_k\,.} \qquad (181)$$

The resulting geodesic equation (179) can be read as

$$\boxed{D\boldsymbol{V}=0\,.} \tag{182}$$

Therefore, (178) can also written as

$$\boxed{d\boldsymbol{V}=d\boldsymbol{V}^{\top}+d\boldsymbol{V}^{\perp}=D\boldsymbol{V}+b_iV^i\mathbf{n}\,,} \tag{183}$$

where $\top$ is the orthogonal projection onto the space spanned by $\{\partial_k\}$ and $\perp$ means the normal component. If there is no normal space $\mathcal{M}^{\perp}$ of $\mathcal{M}$, the differential

$$\boxed{d\boldsymbol{V}=D\boldsymbol{V}} \tag{184}$$

would be the *total* or *absolute* differential of a vector $\boldsymbol{V}$ on surface $\mathcal{M}$. We can multiply $1/du^j$ to the $D\boldsymbol{V}$:

$$\begin{aligned}\frac{1}{du^j}D\boldsymbol{V}&=\frac{1}{du^j}\big(dV^k+\Gamma^k{}_iV^i\big)\partial_k\\&=\frac{1}{du^j}\big(dV^k+\Gamma^k{}_{il}du^lV^i\big)\partial_k\\&=\big(\partial_jV^k+\Gamma^k{}_{il}\underbrace{\frac{du^l}{du^j}}_{\delta^l_j}V^i\big)\partial_k\\&=\big(\partial_jV^k+\Gamma^k{}_{ij}V^i\big)\partial_k\\&:=\big(\nabla_jV^k\big)\partial_k\,,\end{aligned} \tag{185}$$

where we define the component

$$\boxed{\big(D\boldsymbol{V}\big)_j{}^k:=\nabla_j\underbrace{V^k}_{\text{component of }V}=\nabla_{\partial_j}V^k=\partial_jV^k+\Gamma^k{}_{ij}V^i} \tag{186}$$

called the *covariant derivative* of vector V^k in ∂_j direction. We note that the covariant derivatives ∇_j act on the *component* of vector V^k *only*.

It can also be recognized as a vector-valued 1-form $D\boldsymbol{V}$ act on a vector ∂_j

$$D\boldsymbol{V}(\partial_j)=\Big(\big(D\boldsymbol{V}\big)_i{}^kdx^i\otimes\partial_k\Big)(\partial_j)=\big(D\boldsymbol{V}\big)_i{}^k\underbrace{dx^i(\partial_j)}_{\delta^i_j}\otimes\partial_k=\big(D\boldsymbol{V}\big)_j{}^k\partial_k\,. \tag{187}$$

Now we will return to the discussion of the *tangential* part of acceleration (curvature) vector

$$\boldsymbol{a}_{\mathrm{T}}:=\mathbf{p}_t''=\left(\frac{dV}{ds}\right)^{\top}:=\frac{D\boldsymbol{V}}{ds}=\frac{1}{ds}D\boldsymbol{V}=\frac{du^j}{ds}\frac{1}{du^j}D\boldsymbol{V}\overset{(185)}{=}V^j\big(\nabla_jV^k\big)\partial_k:=a^k_{\mathrm{T}}\partial_k\,, \tag{188}$$

which is the orthogonal projection of the acceleration vector $\mathbf{p}''$ onto the space spanned by $\{\partial_k\}$. As a result, we have *tangential* acceleration with component

$$\boxed{a^k_{\mathrm{T}}:=V^j\nabla_jV^k\,,} \tag{189}$$

where

$$V^j\nabla_j=V^j\nabla_{\partial_j}=\nabla_{V^j\partial_j}=\nabla_V\,. \tag{190}$$

So we can also write a_{T}^k as

$$\boxed{a_{\mathrm{T}}^k = \nabla_V V^k\,,} \tag{191}$$

which is called the covariant derivative of V^k along the direction of $\boldsymbol{V}$. Then, we call

$$\boxed{a_{\mathrm{T}}^k = \nabla_V V^k = 0} \quad \text{or} \quad \boxed{D\boldsymbol{V} = 0} \tag{192}$$

the *parallel transport* of tangent vector $\boldsymbol{V}$, which is equivalent to the geodesic equation.

Remark. We would like to remind you the notation of the covariant dereivative in mathematics and physics. Consider a vector V, the covariant dereivative (connection) (D or D_j) of a *full* vector $V = V^i \partial_i$ is

$$DV = du^j \otimes D_j\big(V^i\partial_i\big) = du^j \otimes \big(\underbrace{\partial_j V^i}_{V^i_{,j}} + V^k {\Gamma^i}_{kj}\big)\partial_i := V^i_{;j} du^j \otimes \partial_i = \big(DV\big)_j{}^i du^j \otimes \partial_i\,. \tag{193}$$

We note that here $D_j V^i = \partial_j V^i = V^i_{,j}$ and $D_j \partial_i = \partial_k {\Gamma^k}_{ij}$ as shown in (180). In physics, we always consider a vector represented by its *component* V^i, the covariant dereivative (∇_j) of a vector V^i should be

$$\nabla_j V^i = \big(DV\big)_j{}^i = V^i_{,j} + V^k {\Gamma^i}_{kj} \equiv V^i_{;j}\,. \tag{194}$$

Christoffel symbols According to (123), we can calculate the coefficients Γ_{kij}. Now we would like to derive the coefficients in terms of g_{ij}, the components of first fundamental form $\mathbf{I}$. We take the partial derivative of g_{ij} with respect to u^k and then interchange the indices of the equations. Therefore, we obtain

$$\begin{cases} \dfrac{\partial}{\partial u^k} g_{ij} = \mathbf{p}_{ik} \cdot \mathbf{p}_j + \mathbf{p}_i \cdot \mathbf{p}_{jk} = \Gamma_{jik} + \Gamma_{ijk}\,, & \text{(195a)} \\ \dfrac{\partial}{\partial u^i} g_{jk} = \Gamma_{kji} + \Gamma_{jki}\,, & \text{(195b)} \\ \dfrac{\partial}{\partial u^j} g_{ki} = \Gamma_{ikj} + \Gamma_{kij}\,. & \text{(195c)} \end{cases}$$

Remark. Equations of (195) gives the *metric compatibility*, which can be written as the covarinat deriavative of g_{ij}

$$\nabla_k g_{ij} = \partial_k g_{ij} - {\Gamma^l}_{ik} g_{lj} - {\Gamma^l}_{jk} g_{li} = 0\,. \tag{196}$$

We can define the *non-metricity*

$$Q_{kij} := -\nabla_k g_{ij} \tag{197}$$

and (196) would be equivalent to the vanished non-metricity $Q_{kij} = 0$.

According to (344), which will be shown later that $\mathbf{p}_i \neq \partial_i \mathbf{p}$ in general, we have general case that

$$0 \neq \mathbf{p}_{ij} - \mathbf{p}_{ji} = \big(D_j \mathbf{p}_i + b_{ij}\mathbf{n}\big) - \big(D_i \mathbf{p}_j + b_{ji}\mathbf{n}\big) = \big(\Gamma^k{}_{ij} - \Gamma^k{}_{ji}\big)\mathbf{p}_k + \big(b_{ij} - b_{ji}\big)\mathbf{n}\,. \tag{198}$$

However, in (120), we have $\mathbf{p}_{ij} = \partial_j \mathbf{p}_i = \partial_j \partial_i \mathbf{p}$ due to the globally fixed frame in $\mathbb{E}^3$, which will be explained in the Sec. 4 by the reduction condition (345). Consequently, if we have

$$\mathbf{p}_{ij} - \mathbf{p}_{ji} = 0\,. \quad \Longrightarrow \quad \begin{cases} \Gamma^k{}_{ij} = \Gamma^k{}_{ji}\,, & \text{(199a)} \\ b_{ij} = b_{ji}\,, & \text{(199b)} \end{cases}$$

which give the *symmetric condition* (*torsion-free*) for $\Gamma^k{}_{ij}$ and b_{ij}.

By computing (195c)+(195b)−(195a), we obtain coefficients

$$\boxed{\Gamma_{kij} = \frac{1}{2}\left(\frac{\partial}{\partial u^j} g_{ki} + \frac{\partial}{\partial u^i} g_{jk} - \frac{\partial}{\partial u^k} g_{ij}\right)} \tag{200}$$

and

$$\boxed{\Gamma^k{}_{ij} = g^{kl}\Gamma_{lij} = \frac{1}{2} g^{kl}\left(\frac{\partial}{\partial u^j} g_{li} + \frac{\partial}{\partial u^i} g_{jl} - \frac{\partial}{\partial u^l} g_{ij}\right)\,.} \tag{201}$$

Remark. According to (180b), the $\Gamma^k{}_{ij}$ is also called the *connection coefficients*, or simply the *connection*. Due to *metric compatibility* (196) and *torsion-free* (304a), the connection (200) or (201) is a function of the metric tensor g_{ij}, we also call this kind of connection the *Levi-Civita connection* or *Riemannian connection*. The Levi-Civita connection can also be denoted as

$$\begin{cases} \Gamma_{kij}(g) \equiv [ij,k]\,, & \text{(202a)} \\ \Gamma^k{}_{ij}(g) \equiv \left\{ {k \atop ij} \right\}\,, & \text{(202b)} \end{cases}$$

which are called the *Christoffel symbol* of *first* and *second* kind respectively, in order to distinguish the general connections.

Supplement (Torsion tensor)**.** For general *metric compatible* connection, we always do not have the symmetric property, *i.e.*, $\Gamma^k{}_{ij} \neq \Gamma^k{}_{ji}$. The connection would contain the symmetric and anti-symmetric parts, which is shown as

$$\begin{aligned} \Gamma^k{}_{ij} &= \frac{1}{2}\big(\underbrace{\Gamma^k{}_{ij} + \Gamma^k{}_{ji}}_{\text{symmetric in } i,j}\big) + \frac{1}{2}\big(\underbrace{\Gamma^k{}_{ij} - \Gamma^k{}_{ji}}_{\text{anti-symmetric in } i,j}\big) \\ &= \Gamma^k{}_{(ij)} + \Gamma^k{}_{[ij]}\,. \end{aligned} \tag{203}$$

We define the *torsion tensor*

$$\boxed{T^k{}_{ji} := \Gamma^k{}_{ij} - \Gamma^k{}_{ji} = 2\,\Gamma^k{}_{[ij]}} \tag{204}$$

as the *anti-symmetric* part of the connection. We have to note that $\Gamma^k{}_{(ij)} \neq \left\{ {k \atop ij} \right\}$. The general connection can be decomposed as

$$\boxed{\Gamma^k{}_{ij} = \left\{ {k \atop ij} \right\} + K^k{}_{ij}\,,} \tag{205}$$

and it leads to the relation

$$\boxed{T^k{}_{ji} = K^k{}_{ij} - K^k{}_{ji}\,,} \tag{206}$$

where $K^k{}_{ij}$ is called the *contorsion tensor*. By permutatiing the indices of (206), it can be show that the contorsion $K^k{}_{ij}$ can be in terms of torsion tensor $T^k{}_{ij}$ as

$$\boxed{K^k{}_{ij} := -\frac{1}{2}\left(T^k{}_{ij} + T^i{}_{jk} - T^j{}_{ki}\right) = -\frac{1}{2}\Big(\underbrace{T^k{}_{ij}}_{\text{anti-symmetric in } i,j} + \overbrace{T^i{}_{jk} + T^j{}_{ik}}^{\text{symmetric in } i,j}\Big)} \tag{207}$$

or

$$\boxed{K^k{}_{ij} := -\frac{1}{2}\Big(\underbrace{T^k{}_{ij}}_{\text{anti-symmetric in } i,j} \overbrace{-T_i{}^k{}_j - T_j{}^k{}_i}^{\text{symmetric in } i,j}\Big) \quad (\text{in } \textit{pseudo-Riemannian} \text{ geometry})}\,. \tag{208}$$

So the torsion 2-form can be written as

$$\mathcal{T}^k = \frac{1}{2} T^k{}_{ji} du^j \wedge du^i \overset{(203)}{=} \Gamma^k{}_{ij} du^j \wedge du^i \overset{(205)}{=} K^k{}_{ij} du^j \wedge du^i\,, \tag{209}$$

where we define $K^k{}_i := K^k{}_{ij} du^j$ the contorsion 1-form. Therefore, we have the form equation

$$\boxed{\mathcal{T}^k = K^k{}_i \wedge du^i\,.} \tag{210}$$

Consequently, the symmetric and anti-symmetric parts of the connection are

$$\begin{cases} \Gamma^k{}_{(ij)} = \left\{ {k \atop ij} \right\} + K^k{}_{(ij)}\,, \\ \Gamma^k{}_{[ij]} = K^k{}_{[ij]} = -\frac{1}{2} T^k{}_{ij} = +\frac{1}{2} T^k{}_{ji}\,, \end{cases} \tag{211a}$$

respectively, which shows that the symmetric part $\Gamma^k{}_{(ij)}$ contains Levi-Civita connection $\left\{ {k \atop ij} \right\}$ and torsion $T^k{}_{ij}$.

Remark. We also note that if we identify

$$(\Gamma_a)^c{}_b \equiv \Gamma^c{}_{ba} \tag{212}$$

and the connection form is defined by $\Gamma^k{}_j = \Gamma^k{}_{ij} du^i$, the torsion tensor will be denoted by

$$T^k{}_{ij} := \Gamma^k{}_{ij} - \Gamma^k{}_{ji} = 2\,\Gamma^k{}_{[ij]}\,. \tag{213}$$

Christoffel symbols in the orthogonal coordinates If we consider the case in the *orthogonal* coordinates, we have $g_{12} = g_{21} = g^{12} = g^{21} = 0$ and

$$g^{11} = \frac{1}{g_{11}} = \frac{1}{E}, \quad g^{22} = \frac{1}{g_{22}} = \frac{1}{G}. \tag{214}$$

The component of Christoffel symbols becomes

$${\Gamma^k}_{ij} = \frac{1}{2g_{kk}}\big(\partial_j g_{ki} + \partial_i g_{jk} - \partial_k g_{ij}\big) \quad \text{(no sum)}, \tag{215}$$

and we have the following properties:

- For $j = k$, we have

$${\Gamma^k}_{ik} = \frac{1}{2g_{kk}}\big(\cancel{\partial_k g_{ki}} + \partial_i g_{kk} - \cancel{\partial_k g_{ik}}\big) = \frac{1}{2g_{kk}}\partial_i g_{kk} = \frac{1}{2}\partial_i\big(\ln g_{kk}\big) \quad \text{(no sum)}. \tag{216}$$

- For $i = j \neq k$, we have

$${\Gamma^k}_{ii} = \frac{1}{2g_{kk}}\big(\cancel{\partial_i g_{ki}} + \cancel{\partial_i g_{ik}} - \partial_k g_{ii}\big) = -\frac{1}{2g_{kk}}\partial_k g_{ii} \quad \text{(no sum)}. \tag{217}$$

- In the *general case* of dimension > 2, if $i \neq j \neq k$, we have

$${\Gamma^k}_{ij} = 0 \quad \text{(in } \textit{orthogonal} \text{ coordinates)}. \tag{218}$$

  In dimension $= 2$, it is impossible that i, j, k are all distinct, so we have the same consequence ${\Gamma^k}_{ij} = 0$.

Therefore, for dimension $= 2$, the christoffel symbols in the orthogonal coordinates are given by

$$\begin{aligned}
&{\Gamma^1}_{11} = \frac{E_u}{2E}, \quad {\Gamma^2}_{22} = \frac{G_v}{2G}, \\
&{\Gamma^1}_{12} = {\Gamma^1}_{21} = \frac{E_v}{2E}, \quad {\Gamma^2}_{21} = {\Gamma^2}_{12} = \frac{G_u}{2G}, \\
&{\Gamma^1}_{22} = \frac{-G_u}{2E}, \quad {\Gamma^2}_{11} = \frac{-E_v}{2G}.
\end{aligned} \tag{219}$$

Example (Polar coordinates)**.** Consider the first fundamental form in orthogonal coordinates

$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2 = dr^2 + r^2 d\theta^2, \tag{220}$$

where we have $u = r$, $v = \theta$ and

$$\begin{cases} E = 1, \\ F = 0, \\ G = r^2. \end{cases} \implies \begin{cases} E_r = E_\theta = G_\theta = 0, \\ G_r = 2r. \end{cases} \tag{221}$$

The Christoffel symbols are given by

$${\Gamma^2}_{21} = {\Gamma^2}_{12} = \frac{2r}{2r^2} = \frac{1}{r}, \quad {\Gamma^1}_{22} = \frac{-2r}{2} = -r. \tag{222}$$

The geodesic equations is given by (175). Therefore, each component of the geodesic equations is obtained by

$$\theta'' + {\Gamma^2}_{12} r'\theta' + {\Gamma^2}_{21}\theta' r' = \theta'' + \frac{2}{r}r'\theta' = 0, \tag{223a}$$

$$r'' + {\Gamma^1}_{22}\theta'\theta' = r'' - r\theta'\theta' = 0. \tag{223b}$$

Gauss-Codazzi equation Now we have Gauss and Weingarten formulas

$$\begin{cases} \mathbf{p}_{ik} = \Gamma^l{}_{ik}\mathbf{p}_l + b_{ik}\mathbf{n}\,, & (224a) \\ \mathbf{n}_j = -b^l{}_j\mathbf{p}_l\,, & (224b) \end{cases}$$

which correspond to the derivative vectors of tangent and normal vectors respectively. By taking the partial derivative of $\mathbf{p}_{ik}$ with respect to u^j, we have

$$\begin{aligned} \partial_j\mathbf{p}_{ik} &= \partial_j\Gamma^l{}_{ik}\mathbf{p}_l + \Gamma^l{}_{ik}\mathbf{p}_{lj} + \partial_j b_{ik}\mathbf{n} + b_{ik}\mathbf{n}_j \\ &= \partial_j\Gamma^l{}_{ik}\mathbf{p}_l + \Gamma^l{}_{ik}(\Gamma^m{}_{lj}\mathbf{p}_m + b_{lj}\mathbf{n}) + \partial_j b_{ik}\mathbf{n} + b_{ik}(-b^l{}_j)\mathbf{p}_l \\ &= \big(\partial_j\Gamma^l{}_{ik} + \Gamma^m{}_{ik}\Gamma^l{}_{mj} - b_{ik}b^l{}_j\big)\mathbf{p}_l + \big(\Gamma^l{}_{ik}b_{lj} + \partial_j b_{ik}\big)\mathbf{n}\,. \end{aligned} \tag{225}$$

By interchanging the indices j and k, we obtain

$$\partial_k\mathbf{p}_{ij} = \big(\partial_k\Gamma^l{}_{ij} + \Gamma^m{}_{ij}\Gamma^l{}_{mk} - b_{ij}b^l{}_k\big)\mathbf{p}_l + \big(\Gamma^l{}_{ij}b_{lk} + \partial_k b_{ij}\big)\mathbf{n}\,. \tag{226}$$

As a consequence, (225)−(226)$=0$, which is

$$\partial_j\mathbf{p}_{ik} - \partial_k\mathbf{p}_{ij} = \partial_j\partial_k\mathbf{p}_i - \partial_k\partial_j\mathbf{p}_i = 0\,. \tag{227}$$

We define the *Riemann(-Christoffel) curvature tensor* or simply the *curvature tensor* as

$$\boxed{R^l{}_{ijk} := \partial_j\Gamma^l{}_{ik} - \partial_k\Gamma^l{}_{ij} + \Gamma^l{}_{mj}\Gamma^m{}_{ik} - \Gamma^l{}_{mk}\Gamma^m{}_{ij}\,.} \tag{228}$$

and

$$R_{lijk} := g_{lm}R^m{}_{ijk} \tag{229}$$

is defined. Therefore, according to (227), we obtain a set of equations called *Gauss-Codazzi equation*, which are given by

$$\begin{aligned} 0 &= \partial_j\mathbf{p}_{ik} - \partial_k\mathbf{p}_{ij} \\ &= \big(\underbrace{R^l{}_{ijk} - b_{ik}b^l{}_j + b_{ij}b^l{}_k}_{0}\big)\mathbf{p}_l + \big(\underbrace{\Gamma^l{}_{ik}b_{lj} - \Gamma^l{}_{ij}b_{lk} + \partial_j b_{ik} - \partial_k b_{ij}}_{0}\big)\mathbf{n}\,. \end{aligned} \tag{230}$$

The first one is called *Gauss equation*

$$\boxed{R^l{}_{ijk} = b_{ik}b^l{}_j - b_{ij}b^l{}_k\,,} \tag{231}$$

and the second one is *Codazzi equation*

$$\boxed{\partial_j b_{ik} - \partial_k b_{ij} = \Gamma^l{}_{ij}b_{lk} - \Gamma^l{}_{ik}b_{lj}\,.} \tag{232}$$

We note that there are some symmetries of curvature tensor:

- Anti-symmetric in the indices j and k

$$R^l{}_{ijk} = -R^l{}_{ikj}\,. \tag{233}$$

- Anti-symmetric in l and i (*only* for Levi-Civita connection)

$$R_{lijk} = -R_{iljk}\,. \tag{234}$$

- Symmetric in the pairs of li and jk (*only* for Levi-Civita connection)

$$R_{lijk} = +R_{jkli}\,. \tag{235}$$

We also define the traced curvature tensor called *Ricci tensor*, which is given by

$$\boxed{R_{ik} = R^l{}_{ilk}} \tag{236}$$

and the *scalar curvature* or *Ricci scalar*

$$\boxed{R = g^{ik}R_{ik}\,.} \tag{237}$$

Theorem Egregium of Gauss The indices $i, j, k, l = 1, 2$, by the symmetries of Riemann curvature tensor, the following components are vanished:

$$R_{11jk} = R_{22jk} = 0\,, \quad R_{li11} = R_{li22} = 0\,, \tag{238}$$

From the Gauss equation (231), the residual component can be given by

$$\begin{aligned} R_{1212} &= b_{22}b_{11} - b_{12}b_{12} \\ &= NL - M^2 \\ &= \det(b_{ij}) = \boldsymbol{b}\,. \end{aligned} \tag{239}$$

Therefore, we can rewrite the Gauss curvature (153) as

$$\boxed{K = \frac{\boldsymbol{b}}{\boldsymbol{g}} = \frac{R_{1212}}{\boldsymbol{g}}\,,} \tag{240}$$

which is a function of g_{ij} *only*, *i.e.*, a 2-dimensional surface in $\mathbb{E}^3$ is *totally* determined by it's intrinsic structure. This is the famous *intrinsic geometry of Gauss* and we call this the *theorem Egregium of Gauss*. From the Codazzi equation (232), we only need to consider the case of $i = 1, 2$ and $j = 1$ as well as $k = 2$. This gives

$$\begin{cases} \dfrac{\partial b_{12}}{\partial u^1} - \dfrac{\partial b_{11}}{\partial u^2} = {\Gamma^l}_{11} b_{l2} - {\Gamma^l}_{12} b_{l1}\,, & \text{(241a)} \\[2ex] \dfrac{\partial b_{22}}{\partial u^1} - \dfrac{\partial b_{21}}{\partial u^2} = {\Gamma^l}_{21} b_{l2} - {\Gamma^l}_{22} b_{l1}\,, & \text{(241b)} \end{cases} \qquad (l = 1, 2)\,.$$

> *Remark.* In general n-dimensional space $\mathcal{M}^n$, we can always choose a orthonormal frame, so that $\boldsymbol{g} = 1$. Then we call $K = R_{ijij}$ for $i \neq j$ the *sectional curvature* of the 2-dimensional surface in $\mathcal{M}^n$, where i, j labeled two components on the surface.

Third fundamental (quadratic) form In analogy we have a Gauss map for a surface, the Gauss sphere S^2. A normal vector $\mathbf{n}$ will be sent to be a radius vector of S^2. Therefore, $\mathbf{n}$ on S^2 play the same role as $\mathbf{p}$ on the surface $\mathcal{M}$. As a result, we can calculate the first fundamental for of S^2 by

$$\mathbf{I}|_{S^2} = d\mathbf{n} \cdot d\mathbf{n}\,. \tag{242}$$

Now we define the *third fundamental form* of $\mathcal{M}$ to be the first fundamental form of S^2

$$\mathbf{III}|_{\mathcal{M}} := \mathbf{I}|_{S^2} = d\mathbf{n} \cdot d\mathbf{n}\,. \tag{243}$$

From the first fundamental form, we can calculate the area element on the surface $\mathcal{M}$ and S^2, which are denoted by $\Delta A|_{\mathcal{M}}$ and $\Delta A|_{S^2}$ respectively. The results can be obtained by

$$\begin{cases} \Delta\mathbf{p} \approx \mathbf{p}_u \Delta u + \mathbf{p}_v \Delta v \quad \Longrightarrow \quad \Delta A|_{\mathcal{M}} = |\mathbf{p}_u \Delta u \wedge \mathbf{p}_v \Delta v| = |\mathbf{p}_u \wedge \mathbf{p}_v| \Delta u \Delta v\,, & \text{(244a)} \\ \Delta\mathbf{n} \approx \mathbf{n}_u \Delta u + \mathbf{n}_v \Delta v \quad \Longrightarrow \quad \Delta A|_{S^2} = |\mathbf{n}_u \Delta u \wedge \mathbf{n}_v \Delta v| = |\mathbf{n}_u \wedge \mathbf{n}_v| \Delta u \Delta v\,. & \text{(244b)} \end{cases}$$

However, the vectors $\mathbf{n}_u$ and $\mathbf{n}_v$ are given by the Weingarten formulas (128) with (129) and (130). We can express $\mathbf{n}_u \wedge \mathbf{n}_v$ in terms of $\mathbf{p}_u$ and $\mathbf{p}_v$ by

$$\begin{aligned} \mathbf{n}_u \wedge \mathbf{n}_v &= AD(\mathbf{p}_u \wedge \mathbf{p}_v) - BC(\mathbf{p}_u \wedge \mathbf{p}_v) \\ &= \frac{FM - GL}{EG - F^2}\frac{FM - EN}{EG - F^2}(\mathbf{p}_u \wedge \mathbf{p}_v) - \frac{FL - EM}{EG - F^2}\frac{FN - GM}{EG - F^2}(\mathbf{p}_u \wedge \mathbf{p}_v) \\ &= \frac{LN - M^2}{EG - F^2}(\mathbf{p}_u \wedge \mathbf{p}_v) = \frac{\boldsymbol{b}}{\boldsymbol{g}}(\mathbf{p}_u \wedge \mathbf{p}_v) = K(\mathbf{p}_u \wedge \mathbf{p}_v)\,. \end{aligned} \tag{245}$$

Consequently, we can measure the absolute value of Gauss curvature by the ratio

$$\frac{\Delta A|_{S^2}}{\Delta A|_{\mathcal{M}}} = \frac{|\mathbf{n}_u \wedge \mathbf{n}_v|\Delta u \Delta v}{|\mathbf{p}_u \wedge \mathbf{p}_v|\Delta u \Delta v} = \frac{|K||\mathbf{p}_u \wedge \mathbf{p}_v|\Delta u \Delta v}{|\mathbf{p}_u \wedge \mathbf{p}_v|\Delta u \Delta v} = |K| \,. \tag{246}$$

4 Cartan's moving frame and exterior differentiation methods

We would like to introduce a very useful lemma of Cartan first.

Lemma (Cartan's lemma)**.** *Consider a set of linearly independent frame $\{\boldsymbol{e}_i\}$ (or coframe $\{\vartheta^i\}$) with $i = 1, \ldots, p$ $(p < n)$ in n-dimensional space $\mathcal{M}$ and $\{\boldsymbol{E}_i\}$ is another set of frame. If $\boldsymbol{e}^i \wedge \boldsymbol{E}_i = \boldsymbol{e}^1 \wedge \boldsymbol{E}_1 + \cdots + \boldsymbol{e}^p \wedge \boldsymbol{E}_p = 0$, then $\boldsymbol{E}_i = c_{ij}\boldsymbol{e}^j$ and $c_{ij} = c_{ji}$.*

Proof. We set the linearly independent frame in $\mathcal{M}$ by extending to n-tuple from $\mathbf{e}_i$ given by

$$\underbrace{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_p}_{p}, \underbrace{\mathbf{e}_{p+1}, \ldots, \mathbf{e}_n}_{n-p} \tag{247}$$

with index α labeled components $p+1, p+2, \ldots, n$. We assume that $\boldsymbol{E}_i$ is expanded by frame in $\mathcal{M}$ as

$$\boldsymbol{E}_i = c_{ij}\mathbf{e}^j + c_{i\alpha}\mathbf{e}^\alpha \,. \tag{248}$$

According to $\mathbf{e}^i \wedge \boldsymbol{E}_i = 0$, we have

$$\begin{aligned} 0 = \mathbf{e}^i \wedge \boldsymbol{E}_i &= c_{ij}\mathbf{e}^i \wedge \mathbf{e}^j + c_{i\alpha}\mathbf{e}^i \wedge \mathbf{e}^\alpha \\ &= \frac{1}{2}\big(\underbrace{c_{ij} - c_{ji}}_{0}\big)\mathbf{e}^i \wedge \mathbf{e}^j + \underbrace{c_{i\alpha}}_{0}\mathbf{e}^i \wedge \mathbf{e}^\alpha \,. \end{aligned} \tag{249}$$

Therefore, we obtain the coefficients of $\boldsymbol{E}_i$

$$\begin{cases} c_{ij} = c_{ji} \,, & (250a) \\ c_{i\alpha} = 0 \,, & (250b) \end{cases}$$

which means that $\boldsymbol{E}_i = c_{ij}\mathbf{e}^j$ is constructed by $\{\mathbf{e}_j\}$ *only*. □

Orthonormal frame We have $d\mathbf{p} = \mathbf{p}_i du^i$ with the basis $\mathbf{p}_i$, under the *Gram-Schmit* procedure, we can obtain an orthonormal frame $(\mathbf{p}; \mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}})$ given by (110), (111) and (112), where we use the *hatted indices* to label the component of orthonormal frame now. We can expand $\mathbf{p}_i$ by $\mathbf{e}_{\hat{i}}$ shown as

$$\boxed{\mathbf{p}_i = a^{\hat{j}}{}_i \mathbf{e}_{\hat{j}}} \quad (\hat{\imath}, \hat{\jmath} = \hat{1}, \hat{2}) \quad \text{or} \quad \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} a^{\hat{1}}{}_1 & a^{\hat{2}}{}_1 \\ a^{\hat{1}}{}_2 & a^{\hat{2}}{}_2 \end{pmatrix} \begin{pmatrix} \mathbf{e}_{\hat{1}} \\ \mathbf{e}_{\hat{2}} \end{pmatrix} , \tag{251}$$

where we call the expansion factor $a^{\hat{j}}{}_i$ the *vielbein* (*vierbein* or *tetrad* for 4-dimension), which can be regarded as the $GL(2, \mathbb{R})$ transformation of the frame on $\mathcal{M}$. Therefore, we have to obtain the differential of frame $(\mathbf{p}; \mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}})$. First we can rewrite $d\mathbf{p}$ spanned by frame $\{\mathbf{e}_{\hat{a}}\}$ as

$$\begin{aligned} d\mathbf{p} &= \mathbf{p}_1 du^1 + \mathbf{p}_2 du^2 \\ &= \left(a^{\hat{1}}{}_1\mathbf{e}_{\hat{1}} + a^{\hat{2}}{}_1\mathbf{e}_{\hat{2}}\right)du^1 + \left(a^{\hat{1}}{}_2\mathbf{e}_{\hat{1}} + a^{\hat{2}}{}_2\mathbf{e}_{\hat{2}}\right)du^2 \\ &= \left(a^{\hat{1}}{}_1 du^1 + a^{\hat{1}}{}_2 du^2\right)\mathbf{e}_{\hat{1}} + \left(a^{\hat{2}}{}_1 du^1 + a^{\hat{2}}{}_2 du^2\right)\mathbf{e}_{\hat{2}} := \boxed{\vartheta^{\hat{1}}\mathbf{e}_{\hat{1}} + \vartheta^{\hat{2}}\mathbf{e}_{\hat{2}}} \end{aligned} \tag{252}$$

with

$$\boxed{\vartheta^{\hat{i}} := a^{\hat{i}}{}_j du^j} \tag{253}$$

Then we have to introduce the connection $\omega^{\hat{i}}{}_{\hat{j}}$ for for $\mathbf{e}_{\hat{i}}$. As a result, $d\mathbf{e}_{\hat{i}}$ can be shown as

$$\begin{cases} d\mathbf{e}_{\hat{1}} = \omega^{\hat{1}}{}_{\hat{1}}\mathbf{e}_{\hat{1}} + \omega^{\hat{2}}{}_{\hat{1}}\mathbf{e}_{\hat{2}} + \omega^{\hat{3}}{}_{\hat{1}}\mathbf{e}_{\hat{3}}\,, \\ d\mathbf{e}_{\hat{2}} = \omega^{\hat{1}}{}_{\hat{2}}\mathbf{e}_{\hat{1}} + \omega^{\hat{2}}{}_{\hat{2}}\mathbf{e}_{\hat{2}} + \omega^{\hat{3}}{}_{\hat{2}}\mathbf{e}_{\hat{3}}\,, \\ d\mathbf{e}_{\hat{3}} = \omega^{\hat{1}}{}_{\hat{3}}\mathbf{e}_{\hat{1}} + \omega^{\hat{2}}{}_{\hat{3}}\mathbf{e}_{\hat{2}} + \omega^{\hat{3}}{}_{\hat{3}}\mathbf{e}_{\hat{3}}\,. \end{cases} \quad \Longrightarrow \quad \boxed{d\mathbf{e}_{\hat{a}} = \omega^{\hat{b}}{}_{\hat{a}}\mathbf{e}_{\hat{b}}} \quad (\hat{a}, \hat{b} = \hat{1}, \hat{2}, \hat{3})\,. \tag{254}$$

In particular, we call the connection form $\omega^{\hat{b}}{}_{\hat{a}} = \omega^{\hat{b}}{}_{\hat{a}\hat{c}}\vartheta^{\hat{c}}$ the *linear connection form* and the coefficient $\omega^{\hat{b}}{}_{\hat{a}\hat{c}}$ the *Ricci rotation coefficients* in the *orthonormal* (non-coordinate) frame. However, we have condition for $\omega^{\hat{b}}{}_{\hat{a}}$ due to the orthogonality

$$\mathbf{e}_{\hat{i}} \cdot \mathbf{e}_{\hat{j}} = \delta_{\hat{i}\hat{j}} \quad \text{and} \quad \mathbf{e}_{\hat{a}} \cdot \mathbf{e}_{\hat{3}} = \delta_{\hat{a}\hat{3}}\,. \tag{255}$$

We can differentiate the orthogonality condition $\mathbf{e}_{\hat{i}} \cdot \mathbf{e}_{\hat{j}} = \delta_{\hat{i}\hat{j}}$, thus we have

$$\begin{aligned} d(\mathbf{e}_{\hat{i}} \cdot \mathbf{e}_{\hat{j}}) &= d\mathbf{e}_{\hat{i}} \cdot \mathbf{e}_{\hat{j}} + \mathbf{e}_{\hat{i}} \cdot d\mathbf{e}_{\hat{j}} \\ &= \omega^{\hat{k}}{}_{\hat{i}}\mathbf{e}_{\hat{k}} \cdot \mathbf{e}_{\hat{j}} + \mathbf{e}_{\hat{i}} \cdot \omega^{\hat{k}}{}_{\hat{j}}\mathbf{e}_{\hat{k}} \\ &= \omega^{\hat{k}}{}_{\hat{i}}\delta_{\hat{k}\hat{j}} + \omega^{\hat{k}}{}_{\hat{j}}\delta_{\hat{i}\hat{k}} \\ &= \boxed{\omega^{\hat{j}}{}_{\hat{i}} + \omega^{\hat{i}}{}_{\hat{j}} = 0\,.} \end{aligned} \tag{256}$$

Similarly, we have

$$d(\mathbf{e}_{\hat{i}} \cdot \mathbf{e}_{\hat{3}}) = \boxed{\omega^{\hat{3}}{}_{\hat{i}} + \omega^{\hat{i}}{}_{\hat{3}} = 0} \tag{257}$$

from $\mathbf{e}_{\hat{i}} \cdot \mathbf{e}_{\hat{3}} = \delta_{\hat{i}\hat{3}}$. Therefore, all the components of $\omega^{\hat{b}}{}_{\hat{a}}$ are *anti-symmetric* in the orthonormal frame, we have the consequence:

- *The metric compatibility gives the anti-symmetric property for linear connection form in the orthonormal frame, i.e.,*

$$\nabla_{\hat{c}}\, g_{\hat{a}\hat{b}} = \nabla_{\hat{c}}\, \delta_{\hat{a}\hat{b}} = \underbrace{\mathbf{e}_{\hat{c}}(\delta_{\hat{a}\hat{b}})}_{0} - \omega^{\hat{b}}{}_{\hat{a}\hat{c}} - \omega^{\hat{a}}{}_{\hat{b}\hat{c}} = 0 \quad \Longrightarrow \quad \boxed{\omega^{\hat{b}}{}_{\hat{a}\hat{c}} = -\omega^{\hat{a}}{}_{\hat{b}\hat{c}}\,.} \tag{258}$$

Remark. In *pseudo-Riemannian* space, we have

$$d(\mathbf{e}_{\hat{a}} \cdot \mathbf{e}_{\hat{b}}) = \omega^{\hat{c}}{}_{\hat{a}}\eta_{\hat{c}\hat{b}} + \omega^{\hat{c}}{}_{\hat{b}}\eta_{\hat{a}\hat{c}} = \boxed{\omega_{\hat{b}\hat{a}} + \omega_{\hat{a}\hat{b}} = 0\,,} \tag{259}$$

and the metric compatibility in *pseudo-orthonormal* frame should be read as

$$\nabla_{\hat{c}}\, g_{\hat{a}\hat{b}} = \nabla_{\hat{c}}\, \eta_{\hat{a}\hat{b}} = \underbrace{\mathbf{e}_{\hat{c}}(\eta_{\hat{a}\hat{b}})}_{0} - \omega^{\hat{d}}{}_{\hat{a}\hat{c}}\eta_{\hat{d}\hat{b}} - \omega^{\hat{d}}{}_{\hat{b}\hat{c}}\eta_{\hat{a}\hat{d}} = 0 \quad \Longrightarrow \quad \boxed{\omega_{\hat{b}\hat{a}\hat{c}} = -\omega_{\hat{a}\hat{b}\hat{c}}\,.} \tag{260}$$

Finally the equation (254) is reduced to

$$\begin{cases} d\mathbf{e}_{\hat{1}} = \omega^{\hat{2}}{}_{\hat{1}}\mathbf{e}_{\hat{2}} + \omega^{\hat{3}}{}_{\hat{1}}\mathbf{n}\,, & (261a) \\ d\mathbf{e}_{\hat{2}} = \omega^{\hat{1}}{}_{\hat{2}}\mathbf{e}_{\hat{1}} + \omega^{\hat{3}}{}_{\hat{2}}\mathbf{n}\,, & (261b) \\ d\mathbf{n} = \omega^{\hat{1}}{}_{\hat{3}}\mathbf{e}_{\hat{1}} + \omega^{\hat{2}}{}_{\hat{3}}\mathbf{e}_{\hat{2}}\,. & (261c) \end{cases}$$

Now we can write down the first, second and third fundamental form in orthonormal frame, which are given by

$$\begin{cases} \mathbf{I} = d\mathbf{p}\cdot d\mathbf{p} = \left(\vartheta^{\hat{1}}\right)^2 + \left(\vartheta^{\hat{2}}\right)^2, & (262a) \\ \mathbf{II} = -d\mathbf{p}\cdot d\mathbf{n} = -\vartheta^{\hat{1}}\omega^{\hat{1}}{}_{\hat{3}} - \vartheta^{\hat{2}}\omega^{\hat{2}}{}_{\hat{3}} = \omega^{\hat{3}}{}_{\hat{1}}\vartheta^{\hat{1}} + \omega^{\hat{3}}{}_{\hat{2}}\vartheta^{\hat{2}}\,, & (262b) \\ \mathbf{III} = d\mathbf{n}\cdot d\mathbf{n} = \left(\omega^{\hat{1}}{}_{\hat{3}}\right)^2 + \left(\omega^{\hat{2}}{}_{\hat{3}}\right)^2 = \left(\omega^{\hat{3}}{}_{\hat{1}}\right)^2 + \left(\omega^{\hat{3}}{}_{\hat{2}}\right)^2. & (262c) \end{cases}$$

It can be shown that $\omega^{\hat{3}}{}_{\hat{1}}$ and $\omega^{\hat{3}}{}_{\hat{2}}$ are linear combinations of $\vartheta^{\hat{i}}$ or du^i given by (290) due to *Cartan's first structure equation* (be introduced later) and *Cartan's lemma*

$$\begin{cases} \omega^{\hat{3}}{}_{\hat{1}} = b_{\hat{1}\hat{1}}\vartheta^{\hat{1}} + b_{\hat{1}\hat{2}}\vartheta^{\hat{2}}\,, \\ \omega^{\hat{3}}{}_{\hat{2}} = b_{\hat{2}\hat{1}}\vartheta^{\hat{1}} + b_{\hat{2}\hat{2}}\vartheta^{\hat{2}}\,. \end{cases} \implies \begin{pmatrix} \omega^{\hat{3}}{}_{\hat{1}} \\ \omega^{\hat{3}}{}_{\hat{2}} \end{pmatrix} = \begin{pmatrix} b_{\hat{1}\hat{1}} & b_{\hat{1}\hat{2}} \\ b_{\hat{2}\hat{1}} & b_{\hat{2}\hat{2}} \end{pmatrix} \begin{pmatrix} \vartheta^{\hat{1}} \\ \vartheta^{\hat{2}} \end{pmatrix}. \tag{263}$$

Consequently, the second fundamental form becomes

$$\mathbf{II} = b_{\hat{1}\hat{1}}\left(\vartheta^{\hat{1}}\right)^2 + b_{\hat{1}\hat{2}}\vartheta^{\hat{2}}\vartheta^{\hat{1}} + b_{\hat{2}\hat{1}}\vartheta^{\hat{1}}\vartheta^{\hat{2}} + b_{\hat{2}\hat{2}}\left(\vartheta^{\hat{2}}\right)^2 = b_{\hat{i}\hat{j}}\vartheta^{\hat{i}}\vartheta^{\hat{j}}\,. \tag{264}$$

We consider the following matrix representation for tensors

$$\mathbf{e}_{\hat{i}} \longrightarrow \boldsymbol{e}\,, \quad \vartheta^{\hat{i}} \longrightarrow \boldsymbol{\vartheta}\,, \quad b_{\hat{i}\hat{j}} \longrightarrow \boldsymbol{B}\,. \tag{265}$$

Under the special orthogonal transformation $SO(2,\mathbb{R})$ for frame and coframe, we have a new orthonormal frame

$$\begin{cases} \boldsymbol{e}' = \boldsymbol{P}\boldsymbol{e} \\ \boldsymbol{\vartheta}' = \boldsymbol{P}^{\mathsf{T}}\boldsymbol{\vartheta} \end{cases} \quad \text{with} \quad \boldsymbol{P}^{\mathsf{T}} = \boldsymbol{P}^{-1}\,, \tag{266}$$

where $\boldsymbol{P} \in SO(2,\mathbb{R})$ and T means the *transpose* operation for matrix. As a result, we can obtain the diagonal matrix $\boldsymbol{B}_{\mathrm{D}}$ from $\boldsymbol{B}$ through $\boldsymbol{P}$ by

$$\mathbf{II} = \boldsymbol{\vartheta}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{\vartheta} = (\boldsymbol{\vartheta}')^{\mathsf{T}}\underbrace{\boldsymbol{P}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{P}}_{\boldsymbol{B}_{\mathrm{D}}}\boldsymbol{\vartheta}'\,, \tag{267}$$

i.e.,

$$\boldsymbol{B} \longrightarrow \boldsymbol{B}_{\mathrm{D}} = \boldsymbol{P}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{P} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}. \tag{268}$$

Therefore, the Gauss curvature and mean curvature can be obtained easily by

$$\begin{cases} K = \det(\boldsymbol{B}_{\mathrm{D}}) = \det(\boldsymbol{P}^{\mathsf{T}})\det(\boldsymbol{B})\det(\boldsymbol{P}) = \det(\boldsymbol{B}) = \boxed{b_{\hat{1}\hat{1}}b_{\hat{2}\hat{2}} - b_{\hat{1}\hat{2}}b_{\hat{2}\hat{1}}\,,} & (269a) \\ H = \dfrac{1}{2}\mathrm{tr}\,\boldsymbol{B}_{\mathrm{D}} = \dfrac{1}{2}\mathrm{tr}\,\boldsymbol{B} = \boxed{\dfrac{1}{2}\left(b_{\hat{1}\hat{1}} + b_{\hat{2}\hat{2}}\right),} & (269b) \end{cases}$$

respectively, where the trace of the matrix $\boldsymbol{B}$ is invariant under the $SO(2,\mathbb{R})$ transformation.

Covariant exterior differentiation We define some notation for differential operators for function, vector and 1-form. We use d, $\mathbf{d}$ and $\mathbf{d}_\nabla$ for differentiation, exterior differentiation and *covariant* exterior differentiation respectively.

- For a function (0-form) f, the differential df which can also be regarded as the exterior differentiation of 0-form f:

$$\mathbf{d}_\nabla f = \mathbf{d} f = df\,. \tag{270}$$

- For a vector $\mathbf{e}_{\hat{i}}$, we have an *absolute* differential of vector $d\mathbf{e}_{\hat{i}}$ which is described by Gauss formulas in differential form formalism (126):

$$\mathbf{d}_\nabla \mathbf{e}_{\hat{i}} = d\mathbf{e}_{\hat{i}} = D\mathbf{e}_{\hat{i}} + b_{\hat{i}}\mathbf{n}\,, \tag{271}$$

  is a *vector-valued 1-form*. If there is *no normal space* $\mathcal{M}^\perp$ of $\mathcal{M}$, *i.e.*, there are no $\mathbf{n}$ vector and $b_{\hat{i}\hat{j}}$, the differential is actually equal to the orthogonal projection of vector $\mathbf{e}_{\hat{i}}$ on $\mathcal{M}$

$$\mathbf{d}_\nabla \mathbf{e}_{\hat{i}} = d\mathbf{e}_{\hat{i}} = D\mathbf{e}_{\hat{i}}\,. \tag{272}$$

- For an 1-form $\vartheta^{\hat{i}}$, we only do the exterior differentiation on $\vartheta^{\hat{i}}$:

$$\mathbf{d}_\nabla \vartheta^{\hat{i}} = \mathbf{d}\vartheta^{\hat{i}}\,. \tag{273}$$

Remark. The *covariant exterior differentiation* $\mathbf{d}_\nabla$ is a combined operator, which do the exterior differentiation and covariant derivative on an 1-form and vector respectively.

For a function f, we also note that the second differentiation is

$$d^2 f(x,y) = \frac{\partial^2 f}{\partial x \partial x}dxdx + \frac{\partial^2 f}{\partial y \partial x}dxdy + \frac{\partial^2 f}{\partial x \partial y}dydx + \frac{\partial^2 f}{\partial y \partial y}dydy \neq 0\,, \tag{274}$$

which should not be confused with the second exterior differentiation

$$\mathbf{d}^2 f(x,y) = \mathbf{d}df = \frac{\partial^2 f}{\partial y \partial x}dx \wedge dy + \frac{\partial^2 f}{\partial x \partial y}dy \wedge dx = \left(\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y}\right)dx \wedge dy = 0\,. \tag{275}$$

In addition, $\mathbf{d}_\nabla^2$ would *not* be vanished in general. Therefore, the second fundamental form is

$$\mathrm{II} = -d\mathbf{p}\cdot d\mathbf{n} = +d^2\mathbf{p}\cdot\mathbf{n} = (\mathbf{p}_{ij}du^i du^j)\cdot\mathbf{n} = b_{ij}du^i du^j \tag{276}$$

due to $d\mathbf{p}\cdot\mathbf{n} = 0$ which has been shown in the last term in (120). We note that $d^2\mathbf{p}$ should be realized as a second covariant derivatives of $\mathbf{p}$ in (280).

For coframe du^i, the corresponding exterior differentiation is vanished, which is shown as

$$\mathbf{d}du^i = \mathbf{d}^2 u^i = 0\,. \tag{277}$$

We call $\frac{\partial}{\partial u^i}$ a *holonomic frame* and du^i a *holonomic coframe* which is an *exact form* according to the *Poincaré lemma*. For $\vartheta^{\hat{i}} = a^{\hat{i}}{}_j du^j$, its exterior differentiation is

$$\mathbf{d}\vartheta^{\hat{i}} = \mathbf{d}a^{\hat{i}}{}_j \wedge \mathbf{d}u^j + a^{\hat{i}}{}_j \mathbf{d}^2 u^j \neq 0\,, \tag{278}$$

which is called an *anholonomic coframe* or a *Pfaffian form* dual to the *anholonomic frame* $\mathbf{e}_{\hat{i}}$.

Supplement. We note that the *exterior* 2-form $\mathbf{d}^2_\nabla \mathbf{e}_{\hat{i}}$ will be introduced as the second structure equation through the covariant exterior differentiation and related to the curvature 2-form of (293) and structure constants of (315) later

$$\mathbf{d}^2_\nabla \mathbf{e}_{\hat{i}} = \mathbf{d}_\nabla (d\mathbf{e}_{\hat{i}}) = \frac{1}{2} R^{\hat{l}}{}_{\hat{i}\hat{j}\hat{k}} \vartheta^{\hat{j}} \wedge \vartheta^{\hat{k}} \otimes \mathbf{e}_{\hat{l}} \tag{279a}$$

$$\begin{aligned}
&= \mathbf{d}_\nabla (\vartheta^{\hat{k}} D_{\hat{k}} \mathbf{e}_{\hat{i}}) = \underbrace{\mathbf{d}_\nabla \vartheta^{\hat{k}}}_{(315)} \otimes \underbrace{D_{\hat{k}} \mathbf{e}_{\hat{i}}}_{\Gamma^{\hat{l}}{}_{\hat{i}\hat{k}} \mathbf{e}_{\hat{l}}} - \vartheta^{\hat{k}} \wedge \vartheta^{\hat{j}} \otimes D_{\hat{j}} D_{\hat{k}} \mathbf{e}_{\hat{i}} \\
&= -\frac{1}{2} c^{\hat{k}}{}_{\hat{j}\hat{m}} \vartheta^{\hat{j}} \wedge \vartheta^{\hat{m}} \otimes \Gamma^{\hat{l}}{}_{\hat{i}\hat{k}} \mathbf{e}_{\hat{l}} + \vartheta^{\hat{j}} \wedge \vartheta^{\hat{k}} \otimes \frac{1}{2} \Big(D_{\hat{j}} D_{\hat{k}} - D_{\hat{k}} D_{\hat{j}} \Big) \mathbf{e}_{\hat{i}} \\
&\overset{\hat{k} \leftrightarrow \hat{m}}{=} \vartheta^{\hat{j}} \wedge \vartheta^{\hat{k}} \otimes \frac{1}{2} \Big(- c^{\hat{m}}{}_{\hat{j}\hat{k}} \underbrace{\Gamma^{\hat{l}}{}_{\hat{i}\hat{m}} \mathbf{e}_{\hat{l}}}_{D_{\hat{m}} \mathbf{e}_{\hat{i}}} + \Big(D_{\hat{j}} D_{\hat{k}} - D_{\hat{k}} D_{\hat{j}} \Big) \mathbf{e}_{\hat{i}} \Big)
\end{aligned}$$

$$= \vartheta^{\hat{j}} \wedge \vartheta^{\hat{k}} \otimes \frac{1}{2} \Big(- c^{\hat{m}}{}_{\hat{j}\hat{k}} D_{\hat{m}} + D_{\hat{j}} D_{\hat{k}} - D_{\hat{k}} D_{\hat{j}} \Big) \mathbf{e}_{\hat{i}} \,. \tag{279b}$$

However, the second covariant derivatives of a vector $\mathbf{e}_{\hat{i}}$ should be

$$\begin{aligned}
d^2 \mathbf{e}_{\hat{i}} &= D^2 \mathbf{e}_{\hat{i}} = D(\vartheta^{\hat{k}} \otimes D_{\hat{k}} \mathbf{e}_{\hat{i}}) = \vartheta^{\hat{j}} \otimes \underbrace{D_{\hat{j}} \vartheta^{\hat{k}}}_{\Gamma^{\hat{k}}{}_{\hat{j}\hat{m}} \vartheta^{\hat{m}}} \otimes \underbrace{D_{\hat{k}} \mathbf{e}_{\hat{i}}}_{\Gamma^{\hat{l}}{}_{\hat{i}\hat{k}} \mathbf{e}_{\hat{l}}} + \vartheta^{\hat{j}} \otimes \vartheta^{\hat{k}} \otimes D_{\hat{j}} D_{\hat{k}} \mathbf{e}_{\hat{i}} \\
&\overset{\hat{k} \leftrightarrow \hat{m}}{=} \vartheta^{\hat{j}} \otimes \vartheta^{\hat{k}} \otimes \Big(\Gamma^{\hat{m}}{}_{\hat{j}\hat{k}} \underbrace{\Gamma^{\hat{l}}{}_{\hat{i}\hat{m}} \mathbf{e}_{\hat{l}}}_{D_{\hat{m}} \mathbf{e}_{\hat{i}}} + D_{\hat{j}} D_{\hat{k}} \mathbf{e}_{\hat{i}} \Big)
\end{aligned}$$

$$= \vartheta^{\hat{j}} \otimes \vartheta^{\hat{k}} \otimes \Big(\Gamma^{\hat{m}}{}_{\hat{j}\hat{k}} D_{\hat{m}} + D_{\hat{j}} D_{\hat{k}} \Big) \mathbf{e}_{\hat{i}} \,. \tag{280}$$

Therefor, we conclude that $\mathbf{d}^2_\nabla \mathbf{e}_{\hat{i}} \neq d^2 \mathbf{e}_{\hat{i}}$ because $d\mathbf{e}_{\hat{i}}$ is a *vector-valued 1-form*. We note that that the *wedge product* is obtained by *anti-symmetrizing* the *tensor product*

$$\boxed{A \wedge B = (A \otimes B)^{\mathbf{A}} = \frac{1}{2!} (A \otimes B - B \otimes A)} = \frac{1}{2} (A_{\hat{i}} B_{\hat{j}} - B_{\hat{j}} A_{\hat{i}}) \vartheta^{\hat{i}} \wedge \vartheta^{\hat{j}} \,, \tag{281}$$

where $\mathbf{A}$ indicates the anti-symmetrization. The anti-symmetrization of $D_{\hat{j}} D_{\hat{k}} \mathbf{e}_{\hat{i}}$ and $\Gamma^{\hat{m}}{}_{\hat{j}\hat{k}}$ will be given later, which are shown in (318) and (327b) respectively. As a result, the anti-symmetrization of $d^2 \mathbf{e}_{\hat{i}}$ can be shown as

$$(d^2 \mathbf{e}_{\hat{i}})^{\mathbf{A}} = (D^2 \mathbf{e}_{\hat{i}})^{\mathbf{A}} = D \wedge D\mathbf{e}_{\hat{i}} = \vartheta^{\hat{j}} \wedge \vartheta^{\hat{k}} \otimes \frac{1}{2} \Big(- T^{\hat{m}}{}_{\hat{j}\hat{k}} D_{\hat{m}} + R^{\hat{l}}{}_{\hat{i}\hat{j}\hat{k}} \Big) \mathbf{e}_{\hat{l}} \quad (\textit{cf.}\ (279a) \text{ or } (293)) \,. \tag{282}$$

Canonical 1-form In general case, $\{\mathbf{e}_{\hat{a}}\}$ does not necessarily be chosen as orthonormal, *i.e.*, the metric tensor is $\mathbf{e}_{\hat{a}} \cdot \mathbf{e}_{\hat{b}} = g_{\hat{a}\hat{b}} \neq \delta_{\hat{a}\hat{b}}$. If $\{\mathbf{e}_{\hat{a}}\}$ is an orthonormal frame, we have anti-symmetric property of (256) and (257). We would discuss from the differential of frame $(\mathbf{p}; \mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}})$ and write the equations by the covariant exterior differentiation as

$$\begin{cases} \mathbf{d}_\nabla \mathbf{p} = \vartheta^{\hat{a}} \otimes \mathbf{e}_{\hat{a}} := \vartheta \,, & (283a) \\ \mathbf{d}_\nabla \mathbf{e}_{\hat{a}} = \omega^{\hat{b}}{}_{\hat{a}} \otimes \mathbf{e}_{\hat{b}} \,, & (283b) \end{cases}$$

where we have defined $\vartheta := \mathbf{d}_\nabla \mathbf{p} = d\mathbf{p}$ the *canonical 1-form*, which is a *vector-valued 1-form*. We will show that the canonical 1-form ϑ is an *identity map* of vector in the frame $\mathbf{e}_{\hat{a}}$. Consider a vector $V = V^{\hat{b}}\mathbf{e}_{\hat{b}}$, the canonical 1-form act on V gives

$$\vartheta(V) = \vartheta^{\hat{a}} \otimes \mathbf{e}_{\hat{a}}(V^{\hat{b}}\mathbf{e}_{\hat{b}}) = V^{\hat{b}}\vartheta^{\hat{a}}(\mathbf{e}_{\hat{b}})\mathbf{e}_{\hat{a}} = V^{\hat{b}}\delta^{\hat{a}}_{\hat{b}}\mathbf{e}_{\hat{a}} = V^{\hat{b}}\mathbf{e}_{\hat{b}} = V\,. \tag{284}$$

Remark. If we consider a point $\mathbf{p}$ move on the surface $\mathcal{M}$ in $\mathbb{E}^3$, the differential would be a vector spanned by $\mathbf{e}_{\hat{1}}$ and $\mathbf{e}_{\hat{2}}$ only, which would be written as

$$\mathbf{d}_\nabla \mathbf{p} = \vartheta^{\hat{a}}\mathbf{e}_{\hat{a}} = \vartheta^{\hat{i}}\mathbf{e}_{\hat{i}} = \vartheta^{\hat{1}}\mathbf{e}_{\hat{1}} + \vartheta^{\hat{2}}\mathbf{e}_{\hat{2}} \tag{285}$$

with $\vartheta^{\hat{3}} = 0$, it would be reduced to the equation given by (252).

Cartan's first structure equation Now we do the covariant exterior differentiation on (283). The covariant exterior differentiation of (283a) is

$$\begin{aligned}\mathbf{d}_\nabla\vartheta = \mathbf{d}^2_\nabla\mathbf{p} &= \mathbf{d}_\nabla(\vartheta^{\hat{a}} \otimes \mathbf{e}_{\hat{a}}) \\ &= \mathbf{d}\vartheta^{\hat{a}} \otimes \mathbf{e}_{\hat{a}} + (-1)\,\vartheta^{\hat{a}} \wedge \bar{D}\mathbf{e}_{\hat{a}} \\ &= \mathbf{d}\vartheta^{\hat{a}}\mathbf{e}_{\hat{a}} - \vartheta^{\hat{a}} \wedge \omega^{\hat{b}}{}_{\hat{a}}\mathbf{e}_{\hat{b}} \\ &= \left(\mathbf{d}\vartheta^{\hat{a}} + \omega^{\hat{a}}{}_{\hat{b}} \wedge \vartheta^{\hat{b}}\right)\mathbf{e}_{\hat{a}} \\ &= \left(\mathbf{d}_\nabla\vartheta\right)^{\hat{a}}\mathbf{e}_{\hat{a}} \\ &:= \mathcal{T}^{\hat{a}}\mathbf{e}_{\hat{a}} = \mathcal{T} \neq 0\,, \end{aligned} \tag{286}$$

where $\bar{D}$ is a connection with respect to $\mathbf{e}_{\hat{a}}$. Here we have defined $\mathcal{T}$ the *vector-valued torsion 2-form* and the corresponding component the *torsion 2-form* as

$$\boxed{\mathcal{T}^{\hat{a}} := \left(\mathbf{d}_\nabla\vartheta\right)^{\hat{a}} := \mathbf{D}\underbrace{\vartheta^{\hat{a}}}_{\text{component of }\vartheta} := \mathbf{d}\vartheta^{\hat{a}} + \omega^{\hat{a}}{}_{\hat{b}} \wedge \vartheta^{\hat{b}}\,,} \tag{287}$$

where $\mathbf{D}$ can be identified as operation

$$\boxed{\mathbf{D} = \mathbf{d} + \omega\wedge} \tag{288}$$

act on the differential form which is the *component* of the corresponding vector-valued form. The equation (287) we obtained is called *Cartan's first structure equation*.

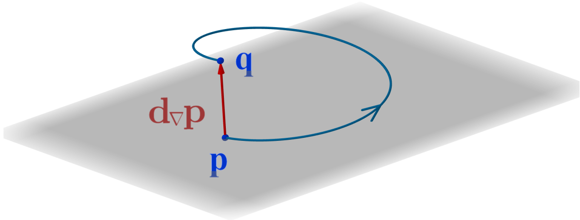
Remark. Since $\mathbf{p}$ moves on the surface $\mathcal{M}$ in $\mathbb{E}^3$, we have $\mathbf{d}_\nabla\mathbf{p} = \vartheta^{\hat{1}}\mathbf{e}_{\hat{1}} + \vartheta^{\hat{2}}\mathbf{e}_{\hat{2}}$ and $\vartheta^{\hat{3}} = 0$. Follow Cartan's first structure equation (287), we have

$$0 = \mathbf{d}_\nabla\vartheta^{\hat{3}} = -\omega^{\hat{3}}{}_{\hat{1}} \wedge \vartheta^{\hat{1}} - \omega^{\hat{3}}{}_{\hat{2}} \wedge \vartheta^{\hat{2}} = -\omega^{\hat{3}}{}_{\hat{i}} \wedge \vartheta^{\hat{i}}\,. \tag{289}$$

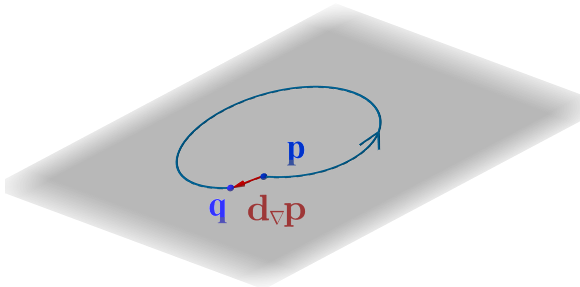
According to *Cartan's lemma*, the connection form is obtained as

$$\omega^{\hat{3}}{}_{\hat{i}} = b_{\hat{i}\hat{j}}\vartheta^{\hat{j}}\,, \tag{290}$$

which gives the equations (263).



(a) Torsion in space.



(b) Torsion on surface.

Figure 8: Torsion is related to the translation.

We can consider an infinitesimal contour integral for $\mathbf{d}_\nabla \mathbf{p}$ infinitesimally around a point as a boundary ∂D of a small region D. By applying *Stokes' theorem* to the contour integral of $\mathbf{d}_\nabla \mathbf{p}$ over ∂D gives

$$\oint_{\partial D} \mathbf{d}_\nabla \mathbf{p} = \int_D \mathbf{d}_\nabla^2 \mathbf{p} = \int_D \mathcal{T} \tag{291}$$

or equivalent to

$$\oint_{\partial D} \vartheta^{\hat{a}} = \int_D \mathbf{D}\vartheta^{\hat{a}} = \int_D \mathcal{T}^{\hat{a}} \,. \tag{292}$$

The integral result implies that the *translation of a point* or the *displacement* $\mathbf{d}_\nabla \mathbf{p}$ is associated with the torsion. If there is no displacement, *i.e.* $\mathbf{d}_\nabla \mathbf{p} = 0$, the space would not be *twisted*.

Cartan's second structure equation Similarly, we do the covariant exterior differentiation on (283b) and obtain

$$\begin{aligned}
\mathbf{d}_\nabla^2 \mathbf{e}_{\hat{a}} &= \mathbf{d}_\nabla (\omega^{\hat{b}}{}_{\hat{a}} \otimes \mathbf{e}_{\hat{b}}) \\
&= \mathbf{d}\omega^{\hat{b}}{}_{\hat{a}} \otimes \mathbf{e}_{\hat{b}} + (-1)\, \omega^{\hat{b}}{}_{\hat{a}} \wedge \bar{D}\mathbf{e}_{\hat{b}} \\
&= \mathbf{d}\omega^{\hat{b}}{}_{\hat{a}} \mathbf{e}_{\hat{b}} - \omega^{\hat{b}}{}_{\hat{a}} \wedge \omega^{\hat{c}}{}_{\hat{b}} \mathbf{e}_{\hat{c}} \\
&= \big(\mathbf{d}\omega^{\hat{b}}{}_{\hat{a}} + \omega^{\hat{b}}{}_{\hat{c}} \wedge \omega^{\hat{c}}{}_{\hat{a}}\big) \mathbf{e}_{\hat{b}} \\
&= \big(\mathbf{d}_\nabla^2 \mathbf{e}_{\hat{a}}\big)^{\hat{b}} \mathbf{e}_{\hat{b}} \\
&:= \mathcal{R}^{\hat{b}}{}_{\hat{a}} \mathbf{e}_{\hat{b}} = \mathcal{R}_{\hat{a}} \neq 0 \,.
\end{aligned} \tag{293}$$

Therefore we have the *vector-valued curvature 2-form* $\mathcal{R}_{\hat{a}}$ with the corresponding component *curvature 2-form* given by

$$\boxed{\mathcal{R}^{\hat{b}}{}_{\hat{a}} := \big(\mathbf{d}_\nabla^2 \mathbf{e}_{\hat{a}}\big)^{\hat{b}} := \mathbf{D} \underbrace{\omega^{\hat{b}}{}_{\hat{a}}}_{\text{component of } \mathbf{d}_\nabla \mathbf{e}_{\hat{a}}} := \mathbf{d}\omega^{\hat{b}}{}_{\hat{a}} + \omega^{\hat{b}}{}_{\hat{c}} \wedge \omega^{\hat{c}}{}_{\hat{a}} \,,} \tag{294}$$

which is called *Cartan's second structure equation*.

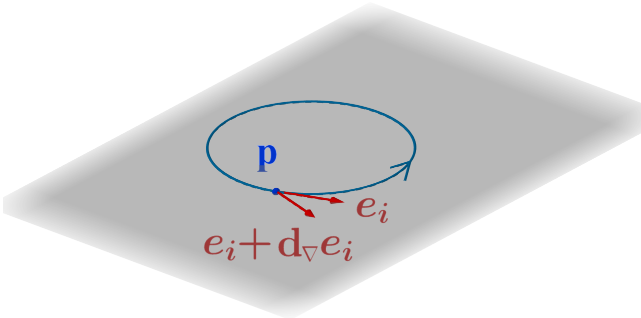


Figure 9: Curvature is related to the rotation.

The similar infinitesimal contour integral for $\mathbf{d}_\nabla \mathbf{e}_{\hat{a}}$ gives

$$\oint_{\partial D} \mathbf{d}_\nabla \mathbf{e}_{\hat{a}} = \int_D \mathbf{d}^2_\nabla \mathbf{e}_{\hat{a}} = \int_D \mathcal{R}_{\hat{a}} \tag{295}$$

or equivalent to

$$\oint_{\partial D} \omega^{\hat{b}}{}_{\hat{a}} = \int_D \mathbf{D}\omega^{\hat{b}}{}_{\hat{a}} = \int_D \mathcal{R}^{\hat{b}}{}_{\hat{a}} \,, \tag{296}$$

which means that the *rotation of a vector* is associated with the curvature. If the vector does not change the direction after moving around a contour, *i.e.* $\mathbf{d}_\nabla \mathbf{e}_{\hat{a}} = 0$, the space would be *flat*.

First Bianchi identity The exterior differentiation of two structure equations can get more information of torsion and curvature. The structure equations are

$$\begin{cases} \mathcal{T}^{\hat{a}} = \mathbf{d}\vartheta^{\hat{a}} + \omega^{\hat{a}}{}_{\hat{b}} \wedge \vartheta^{\hat{b}} \,, & \text{(297a)} \\ \mathcal{R}^{\hat{a}}{}_{\hat{b}} = \mathbf{d}\omega^{\hat{a}}{}_{\hat{b}} + \omega^{\hat{a}}{}_{\hat{c}} \wedge \omega^{\hat{c}}{}_{\hat{b}} \,. & \text{(297b)} \end{cases}$$

We take the exterior differentiation of the first structure equation shown by

$$\begin{aligned} \mathbf{d}\mathcal{T}^{\hat{a}} &= \mathbf{d}^2\vartheta^{\hat{a}} + \mathbf{d}\omega^{\hat{a}}{}_{\hat{b}} \wedge \vartheta^{\hat{b}} - \omega^{\hat{a}}{}_{\hat{b}} \wedge \mathbf{d}\vartheta^{\hat{b}} \\ &= \left(\mathcal{R}^{\hat{a}}{}_{\hat{b}} - \omega^{\hat{a}}{}_{\hat{c}} \wedge \omega^{\hat{c}}{}_{\hat{b}}\right) \wedge \vartheta^{\hat{b}} - \omega^{\hat{a}}{}_{\hat{b}} \wedge \left(\mathcal{T}^{\hat{b}} - \omega^{\hat{b}}{}_{\hat{c}} \wedge \vartheta^{\hat{c}}\right) \\ &= \mathcal{R}^{\hat{a}}{}_{\hat{b}} \wedge \vartheta^{\hat{b}} - \cancel{\omega^{\hat{a}}{}_{\hat{c}} \wedge \omega^{\hat{c}}{}_{\hat{b}} \wedge \vartheta^{\hat{b}}} - \omega^{\hat{a}}{}_{\hat{b}} \wedge \mathcal{T}^{\hat{b}} + \cancel{\omega^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}}{}_{\hat{c}} \wedge \vartheta^{\hat{c}}} \,, \end{aligned} \tag{298}$$

then we obtain the *first Bianchi identity*

$$\boxed{\mathbf{D}\mathcal{T}^{\hat{a}} = \mathbf{d}\mathcal{T}^{\hat{a}} + \omega^{\hat{a}}{}_{\hat{b}} \wedge \mathcal{T}^{\hat{b}} = \mathcal{R}^{\hat{a}}{}_{\hat{b}} \wedge \vartheta^{\hat{b}} \,.} \tag{299}$$

If we have torsion-free condition $\mathcal{T}^{\hat{a}} = 0$, the first Bianchi identity becomes

$$\begin{aligned} 0 &= \mathcal{R}^{\hat{a}}{}_{\hat{b}} \wedge \vartheta^{\hat{b}} \\ &= \frac{1}{2} R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} \vartheta^{\hat{c}} \wedge \vartheta^{\hat{d}} \wedge \vartheta^{\hat{b}} \\ &= \frac{1}{2}\left(R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} + R^{\hat{a}}{}_{\hat{c}\hat{d}\hat{b}} + R^{\hat{a}}{}_{\hat{d}\hat{b}\hat{c}} - R^{\hat{a}}{}_{\hat{b}\hat{d}\hat{c}} - R^{\hat{a}}{}_{\hat{c}\hat{b}\hat{d}} - R^{\hat{a}}{}_{\hat{d}\hat{c}\hat{b}}\right) \vartheta^{\hat{c}} \wedge \vartheta^{\hat{d}} \wedge \vartheta^{\hat{b}} \\ &= \left(R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} + R^{\hat{a}}{}_{\hat{c}\hat{d}\hat{b}} + R^{\hat{a}}{}_{\hat{d}\hat{b}\hat{c}}\right) \vartheta^{\hat{c}} \wedge \vartheta^{\hat{d}} \wedge \vartheta^{\hat{b}} \,, \end{aligned} \tag{300}$$

resulting in

$$\boxed{R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} + R^{\hat{a}}{}_{\hat{c}\hat{d}\hat{b}} + R^{\hat{a}}{}_{\hat{d}\hat{b}\hat{c}} = 0\,.} \tag{301}$$

Second Bianchi identity Similarly, the exterior differentiation of the second structure equation is

$$\begin{aligned}
\mathbf{d}\mathcal{R}^{\hat{a}}{}_{\hat{b}} &= \mathbf{d}^2\omega^{\hat{a}}{}_{\hat{b}} + \mathbf{d}\omega^{\hat{a}}{}_{\hat{c}} \wedge \omega^{\hat{c}}{}_{\hat{b}} - \omega^{\hat{a}}{}_{\hat{c}} \wedge \mathbf{d}\omega^{\hat{c}}{}_{\hat{b}} \\
&= \left(\mathcal{R}^{\hat{a}}{}_{\hat{c}} - \omega^{\hat{a}}{}_{\hat{d}} \wedge \omega^{\hat{d}}{}_{\hat{c}}\right) \wedge \omega^{\hat{c}}{}_{\hat{b}} - \omega^{\hat{a}}{}_{\hat{c}} \wedge \left(\mathcal{R}^{\hat{c}}{}_{\hat{b}} - \omega^{\hat{c}}{}_{\hat{d}} \wedge \omega^{\hat{d}}{}_{\hat{b}}\right) \\
&= \underbrace{\mathcal{R}^{\hat{a}}{}_{\hat{c}} \wedge \omega^{\hat{c}}{}_{\hat{b}}}_{+\omega^{\hat{c}}{}_{\hat{b}} \wedge \mathcal{R}^{\hat{a}}{}_{\hat{c}} \text{ due to 2-form } \mathcal{R}^{\hat{a}}{}_{\hat{c}}} \cancel{- \omega^{\hat{a}}{}_{\hat{d}} \wedge \omega^{\hat{d}}{}_{\hat{c}} \wedge \omega^{\hat{c}}{}_{\hat{b}}} - \omega^{\hat{a}}{}_{\hat{c}} \wedge \mathcal{R}^{\hat{c}}{}_{\hat{b}} \cancel{+ \omega^{\hat{a}}{}_{\hat{c}} \wedge \omega^{\hat{c}}{}_{\hat{d}} \wedge \omega^{\hat{d}}{}_{\hat{b}}}\,,
\end{aligned} \tag{302}$$

which leads to the *second Bianchi identity*

$$\boxed{\mathbf{D}\mathcal{R}^{\hat{a}}{}_{\hat{b}} = \mathbf{d}\mathcal{R}^{\hat{a}}{}_{\hat{b}} - \omega^{\hat{c}}{}_{\hat{b}} \wedge \mathcal{R}^{\hat{a}}{}_{\hat{c}} + \omega^{\hat{a}}{}_{\hat{c}} \wedge \mathcal{R}^{\hat{c}}{}_{\hat{b}} = 0\,.} \tag{303}$$

Remark. It is essential to consider the geometric structure from Cartan's viewpoint. The first structure equations in Riemannian geometry is restricted to be torsion-free condition. Then the structure equations are reduced to

$$\begin{cases}
\mathbf{d}\vartheta^{\hat{a}} = -\omega^{\hat{a}}{}_{\hat{b}} \wedge \vartheta^{\hat{b}} = +\vartheta^{\hat{b}} \wedge \omega^{\hat{a}}{}_{\hat{b}}\,, & \text{(304a)} \\
\mathcal{R}^{\hat{a}}{}_{\hat{b}} = \mathbf{d}\omega^{\hat{a}}{}_{\hat{b}} + \omega^{\hat{a}}{}_{\hat{c}} \wedge \omega^{\hat{c}}{}_{\hat{b}}\,. & \text{(304b)}
\end{cases}$$

Because of metric compatibility, we have $\omega^{\hat{a}}{}_{\hat{b}} = -\omega^{\hat{b}}{}_{\hat{a}}$ (or $\omega_{\hat{a}\hat{b}} = -\omega_{\hat{b}\hat{a}}$ in *pseudo-Riemannian* geometry), which gives

$$\omega^{\hat{a}}{}_{\hat{b}} = \omega^{\hat{a}}{}_{\hat{b}\hat{c}}\vartheta^{\hat{c}} = -\omega^{\hat{b}}{}_{\hat{a}} = -\omega^{\hat{b}}{}_{\hat{a}\hat{c}}\vartheta^{\hat{c}} \quad \Longrightarrow \quad \omega^{\hat{a}}{}_{\hat{b}\hat{c}} = -\omega^{\hat{b}}{}_{\hat{a}\hat{c}}\,. \tag{305}$$

Due to (304a), we obtain

$$\mathbf{d}\vartheta^{\hat{a}} = \vartheta^{\hat{b}} \wedge \omega^{\hat{a}}{}_{\hat{b}} = \omega^{\hat{a}}{}_{\hat{b}\hat{c}}\vartheta^{\hat{b}} \wedge \vartheta^{\hat{c}} = \frac{1}{2}\left(\omega^{\hat{a}}{}_{\hat{b}\hat{c}} - \omega^{\hat{a}}{}_{\hat{c}\hat{b}}\right)\vartheta^{\hat{b}} \wedge \vartheta^{\hat{c}}\,. \tag{306}$$

However, $\mathbf{d}\vartheta^{\hat{a}}$ is a 2-form, it can be written as

$$\mathbf{d}\vartheta^{\hat{a}} = \frac{1}{2}a^{\hat{a}}{}_{\hat{b}\hat{c}}\vartheta^{\hat{b}} \wedge \vartheta^{\hat{c}}\,, \tag{307}$$

which leads to

$$a^{\hat{a}}{}_{\hat{b}\hat{c}} = \omega^{\hat{a}}{}_{\hat{b}\hat{c}} - \omega^{\hat{a}}{}_{\hat{c}\hat{b}}\,. \tag{308}$$

By permutating the indices $\hat{a}$, $\hat{b}$ and $\hat{c}$, we have the equation

$$a^{\hat{a}}{}_{\hat{b}\hat{c}} + a^{\hat{b}}{}_{\hat{c}\hat{a}} - a^{\hat{c}}{}_{\hat{a}\hat{b}} = \omega^{\hat{a}}{}_{\hat{b}\hat{c}} - \omega^{\hat{b}}{}_{\hat{a}\hat{c}} = 2\omega^{\hat{a}}{}_{\hat{b}\hat{c}}\,. \tag{309}$$

The resulting connection coefficients are

$$\omega^{\hat{a}}{}_{\hat{b}\hat{c}} = \frac{1}{2}\left(a^{\hat{a}}{}_{\hat{b}\hat{c}} + a^{\hat{b}}{}_{\hat{c}\hat{a}} - a^{\hat{c}}{}_{\hat{a}\hat{b}}\right)\,. \tag{310}$$

It can be shown that

$$\boxed{a^{\hat{a}}{}_{\hat{b}\hat{c}} = -c^{\hat{a}}{}_{\hat{b}\hat{c}}\,,} \tag{311}$$

where $c^{\hat{a}}{}_{\hat{b}\hat{c}}$ is called the *structure constants* or *commutation coefficients*, which is defined by the commutation relation of the anholonomic frame

$$\begin{aligned}
[\mathbf{e}_{\hat{a}}, \mathbf{e}_{\hat{b}}] &= \left[a_{\hat{a}}{}^{a}\frac{\partial}{\partial x^{a}}, a_{\hat{b}}{}^{b}\frac{\partial}{\partial x^{b}}\right] \\
&= a_{\hat{a}}{}^{a}\frac{\partial}{\partial x^{a}}\left(a_{\hat{b}}{}^{b}\frac{\partial}{\partial x^{b}}\right) - a_{\hat{b}}{}^{b}\frac{\partial}{\partial x^{b}}\left(a_{\hat{a}}{}^{a}\frac{\partial}{\partial x^{a}}\right) \\
&= a_{\hat{a}}{}^{a}\frac{\partial}{\partial x^{a}}\left(a_{\hat{b}}{}^{b}\right)\frac{\partial}{\partial x^{b}} - a_{\hat{b}}{}^{b}\frac{\partial}{\partial x^{b}}\left(a_{\hat{a}}{}^{a}\right)\frac{\partial}{\partial x^{a}} \\
&= a_{\hat{a}}{}^{a}\frac{\partial}{\partial x^{a}}\left(a_{\hat{b}}{}^{b}\right)a^{\hat{c}}{}_{b}\mathbf{e}_{\hat{c}} - a_{\hat{b}}{}^{b}\frac{\partial}{\partial x^{b}}\left(a_{\hat{a}}{}^{a}\right)a^{\hat{c}}{}_{a}\mathbf{e}_{\hat{c}} \\
&= \left[a_{\hat{a}}{}^{a}\frac{\partial}{\partial x^{a}}\left(a_{\hat{b}}{}^{b}\right)a^{\hat{c}}{}_{b} - a_{\hat{b}}{}^{b}\frac{\partial}{\partial x^{b}}\left(a_{\hat{a}}{}^{a}\right)a^{\hat{c}}{}_{a}\right]\mathbf{e}_{\hat{c}} \\
&= \left[\mathbf{e}_{\hat{a}}\left(a_{\hat{b}}{}^{b}\right)a^{\hat{c}}{}_{b} - \mathbf{e}_{\hat{b}}\left(a_{\hat{a}}{}^{a}\right)a^{\hat{c}}{}_{a}\right]\mathbf{e}_{\hat{c}} := c^{\hat{c}}{}_{\hat{a}\hat{b}}\mathbf{e}_{\hat{c}}\,.
\end{aligned} \tag{312}$$

Here the anholonomic frame $\mathbf{e}_{\hat{a}}$ is identified as the so-called *Pfaffian derivative*. As a result, we obtain the the *structure constants*

$$\boxed{c^{\hat{c}}{}_{\hat{a}\hat{b}} := \mathbf{e}_{\hat{a}}\left(a_{\hat{b}}{}^{b}\right)a^{\hat{c}}{}_{b} - \mathbf{e}_{\hat{b}}\left(a_{\hat{a}}{}^{a}\right)a^{\hat{c}}{}_{a}} \tag{313}$$

We note that it is apparent that the commutator

$$[\partial_a, \partial_b] = 0 \tag{314}$$

because two partial derivatives can be interchanged. As a result, we conclude that there is no structure constants in holonomic frame. Therefore, the commutation coefficients can also be called *anholonomity* which characterizes the property of the anholonomic frame. On the other hand,

$$\begin{aligned}
\mathbf{d}\vartheta^{\hat{c}} &= \mathbf{d}\left(a^{\hat{c}}{}_{b}dx^{b}\right) \\
&= d\left(a^{\hat{c}}{}_{b}\right)dx^{b} \\
&= \left(\frac{\partial}{\partial x^{a}}a^{\hat{c}}{}_{b}\right)dx^{a}\wedge dx^{b} \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x^{a}}a^{\hat{c}}{}_{b} - \frac{\partial}{\partial x^{b}}a^{\hat{c}}{}_{a}\right)\left(a_{\hat{a}}{}^{a}\vartheta^{\hat{a}}\right)\wedge\left(a_{\hat{b}}{}^{b}\vartheta^{\hat{b}}\right) \\
&= \frac{1}{2}\left(a_{\hat{b}}{}^{b}a_{\hat{a}}{}^{a}\frac{\partial}{\partial x^{a}}a^{\hat{c}}{}_{b} - a_{\hat{a}}{}^{a}a_{\hat{b}}{}^{b}\frac{\partial}{\partial x^{b}}a^{\hat{c}}{}_{a}\right)\vartheta^{\hat{a}}\wedge\vartheta^{\hat{b}} \\
&= \frac{1}{2}\left(a_{\hat{b}}{}^{b}\mathbf{e}_{\hat{a}}(a^{\hat{c}}{}_{b}) - a_{\hat{a}}{}^{a}\mathbf{e}_{\hat{b}}(a^{\hat{c}}{}_{a})\right)\vartheta^{\hat{a}}\wedge\vartheta^{\hat{b}} \\
&= -\frac{1}{2}\left(a^{\hat{c}}{}_{b}\mathbf{e}_{\hat{a}}(a_{\hat{b}}{}^{b}) - a^{\hat{c}}{}_{a}\mathbf{e}_{\hat{b}}(a_{\hat{a}}{}^{a})\right)\vartheta^{\hat{a}}\wedge\vartheta^{\hat{b}} \\
&= -\frac{1}{2}c^{\hat{c}}{}_{\hat{a}\hat{b}}\vartheta^{\hat{a}}\wedge\vartheta^{\hat{b}}\,,
\end{aligned} \tag{315}$$

where we have used $a_{\hat{b}}{}^{b}\mathbf{e}_{\hat{a}}(a^{\hat{c}}{}_{b}) = -a^{\hat{c}}{}_{b}\mathbf{e}_{\hat{a}}(a_{\hat{b}}{}^{b})$ due to $a_{\hat{b}}{}^{b}a^{\hat{c}}{}_{b} = \delta^{\hat{c}}_{\hat{b}}$. The above result proves (311) and finally we obtain the linear connection coefficients

$$\boxed{\omega^{\hat{a}}{}_{\hat{b}\hat{c}} = -\frac{1}{2}\left(c^{\hat{a}}{}_{\hat{b}\hat{c}} + c^{\hat{b}}{}_{\hat{c}\hat{a}} - c^{\hat{c}}{}_{\hat{a}\hat{b}}\right)} \tag{316}$$

or

$$\boxed{\omega^{\hat{a}}{}_{\hat{b}\hat{c}} = -\frac{1}{2}\left(c^{\hat{a}}{}_{\hat{b}\hat{c}} - c_{\hat{b}}{}^{\hat{a}}{}_{\hat{c}} - c_{\hat{c}}{}^{\hat{a}}{}_{\hat{b}}\right) \quad (\text{in } \textit{pseudo-Riemannian} \text{ geometry})\,.} \tag{317}$$

Supplement (Covariant derivative in anholonomic frame)**.** We can do the calculation in both holonomic and anholonomic frame. However, according to (251), we have

$$\begin{aligned}
D_{\hat{j}}D_{\hat{k}}\mathbf{e}_{\hat{i}} &= a_{\hat{j}}{}^{j}D_{j}\left(a_{\hat{k}}{}^{k}D_{k}\mathbf{e}_{\hat{i}}\right)\\
&= a_{\hat{j}}{}^{j}a_{\hat{k}}{}^{k}D_{j}D_{k}\mathbf{e}_{\hat{i}} + a_{\hat{j}}{}^{j}(\partial_{j}a_{\hat{k}}{}^{k})(D_{k}\mathbf{e}_{\hat{i}})\\
&= a_{\hat{j}}{}^{j}a_{\hat{k}}{}^{k}D_{j}D_{k}(a_{\hat{i}}{}^{i}\mathbf{p}_{i}) + a^{\hat{l}}{}_{k}(\mathbf{e}_{\hat{j}}a_{\hat{k}}{}^{k})(D_{\hat{l}}\mathbf{e}_{\hat{i}})\\
&= a_{\hat{j}}{}^{j}a_{\hat{k}}{}^{k}D_{j}\left(a_{\hat{i}}{}^{i}D_{k}\mathbf{p}_{i} + (\partial_{k}a_{\hat{i}}{}^{i})\mathbf{p}_{i}\right) + a^{\hat{l}}{}_{k}(\mathbf{e}_{\hat{j}}a_{\hat{k}}{}^{k})(D_{\hat{l}}\mathbf{e}_{\hat{i}})\\
&= a_{\hat{j}}{}^{j}a_{\hat{k}}{}^{k}\left(a_{\hat{i}}{}^{i}D_{j}D_{k}\mathbf{p}_{i} + (\partial_{j}a_{\hat{i}}{}^{i})D_{k}\mathbf{p}_{i} + (\partial_{k}a_{\hat{i}}{}^{i})D_{j}\mathbf{p}_{i} + (\partial_{j}\partial_{k}a_{\hat{i}}{}^{i})\mathbf{p}_{i}\right)\\
&\quad + a^{\hat{l}}{}_{k}(\mathbf{e}_{\hat{j}}a_{\hat{k}}{}^{k})(D_{\hat{l}}\mathbf{e}_{\hat{i}})\,.
\end{aligned} \tag{318}$$

So we can find

$$\left(D_{\hat{j}}D_{\hat{k}} - D_{\hat{k}}D_{\hat{j}}\right)\mathbf{e}_{\hat{i}} = a_{\hat{i}}{}^{i}a_{\hat{j}}{}^{j}a_{\hat{k}}{}^{k}\left(D_{j}D_{k} - D_{k}D_{j}\right)\mathbf{p}_{i} + \left(a^{\hat{l}}{}_{k}(\mathbf{e}_{\hat{j}}a_{\hat{k}}{}^{k}) - a^{\hat{l}}{}_{k}(\mathbf{e}_{\hat{k}}a_{\hat{j}}{}^{k})\right)(D_{\hat{l}}\mathbf{e}_{\hat{i}})\,. \tag{319}$$

Here we move the last term of (319) to the left-handed side and use the structure constants $c^{\hat{l}}{}_{\hat{j}\hat{k}}$ defined by (313). Then, we also use (190) and (312) to obtain the equation

$$\begin{aligned}
a_{\hat{i}}{}^{i}a_{\hat{j}}{}^{j}a_{\hat{k}}{}^{k}(D_{j}D_{k} - D_{k}D_{j})\mathbf{p}_{i} &= \left(D_{\hat{j}}D_{\hat{k}} - D_{\hat{k}}D_{\hat{j}} - c^{\hat{l}}{}_{\hat{j}\hat{k}}D_{\hat{l}}\right)\mathbf{e}_{\hat{i}}\\
&= \left(D_{\hat{j}}D_{\hat{k}} - D_{\hat{k}}D_{\hat{j}} - D_{c^{\hat{l}}{}_{\hat{j}\hat{k}}\mathbf{e}_{\hat{l}}}\right)\mathbf{e}_{\hat{i}}\\
&= \left(D_{\hat{j}}D_{\hat{k}} - D_{\hat{k}}D_{\hat{j}} - D_{[\mathbf{e}_{\hat{j}},\mathbf{e}_{\hat{k}}]}\right)\mathbf{e}_{\hat{i}}\,.
\end{aligned} \tag{320}$$

It gives the general formula of curvature tensor

$$\text{Holonomic frame:} \quad R^{l}{}_{ijk}\mathbf{p}_{l} = \left(D_{j}D_{k} - D_{k}D_{j} - \underbrace{0}_{D_{[\partial_{j},\partial_{k}]}=0 \text{ which is vanished due to (314).}}\right)\mathbf{p}_{i}\,, \tag{321a}$$

$$\text{Anholonomic frame:} \quad R^{\hat{l}}{}_{\hat{i}\hat{j}\hat{k}}\mathbf{e}_{\hat{l}} = \left(D_{\hat{j}}D_{\hat{k}} - D_{\hat{k}}D_{\hat{j}} - D_{[\mathbf{e}_{\hat{j}},\mathbf{e}_{\hat{k}}]}\right)\mathbf{e}_{\hat{i}}\,. \tag{321b}$$

Therefore, one can consider three vectors $X = X^{\hat{j}}\mathbf{e}_{\hat{j}}$, $Y = Y^{\hat{k}}\mathbf{e}_{\hat{k}}$ and $Z = Z^{\hat{i}}\mathbf{e}_{\hat{i}}$, then it can be shown that the frame independent formula of curvature tensor is

$$\boxed{X^{\hat{j}}Y^{\hat{k}}Z^{\hat{i}}R^{\hat{l}}{}_{\hat{i}\hat{j}\hat{k}}\mathbf{e}_{\hat{l}} = \left(D_{X}D_{Y} - D_{Y}D_{X} - D_{[X,Y]}\right)Z := R(X,Y)Z\,.} \tag{322}$$

After the calculation, (321) can be written in terms of the connections and structure constants

$$\text{Holonomic:} \quad \boxed{R^{l}{}_{ijk} = \partial_{j}\Gamma^{l}{}_{ik} - \partial_{k}\Gamma^{l}{}_{ij} + \Gamma^{l}{}_{mj}\Gamma^{m}{}_{ik} - \Gamma^{l}{}_{mk}\Gamma^{m}{}_{ij}\,,} \tag{323a}$$

$$\text{Anholonomic:} \quad \boxed{R^{\hat{l}}{}_{\hat{i}\hat{j}\hat{k}} = \mathbf{e}_{\hat{j}}\omega^{\hat{l}}{}_{\hat{i}\hat{k}} - \mathbf{e}_{\hat{k}}\omega^{\hat{l}}{}_{\hat{i}\hat{j}} + \omega^{\hat{l}}{}_{\hat{m}\hat{j}}\omega^{\hat{m}}{}_{\hat{i}\hat{k}} - \omega^{\hat{l}}{}_{\hat{m}\hat{k}}\omega^{\hat{m}}{}_{\hat{i}\hat{j}} - \omega^{\hat{l}}{}_{\hat{i}\hat{m}}c^{\hat{m}}{}_{\hat{j}\hat{k}}\,.} \tag{323b}$$

Finally (320) gives the transformation formula for curvature tensor by substituting $\mathbf{p}_l = a^{\hat{l}}{}_l \mathbf{e}_{\hat{l}}$ to the left-handed side of (320)

$$R^{\hat{l}}{}_{\hat{i}\hat{j}\hat{k}} = a^{\hat{l}}{}_l a_{\hat{i}}{}^i a_{\hat{j}}{}^j a_{\hat{k}}{}^k R^l{}_{ijk} \,. \tag{324}$$

Similarly, it can be shown that by computing the commutator of the covariant derivatives on function f, *i.e.*, $\big(D_{\hat{j}} D_{\hat{k}} - D_{\hat{k}} D_{\hat{j}}\big) f$, we obtain the following equations

$$\begin{cases} \text{Holonomic frame:} & T^i{}_{jk} D_i f = T^i{}_{jk} \partial_i f = \big(D_j D_k - D_k D_j\big) f \,, & (325\text{a}) \\ \text{Anholonomic frame:} & T^{\hat{i}}{}_{\hat{j}\hat{k}} D_{\hat{i}} f = T^{\hat{i}}{}_{\hat{j}\hat{k}} \mathbf{e}_{\hat{i}} f = \big(D_{\hat{j}} D_{\hat{k}} - D_{\hat{k}} D_{\hat{j}} - [\mathbf{e}_{\hat{j}}, \mathbf{e}_{\hat{k}}]\big) f \,, & (325\text{b}) \end{cases}$$

and the frame independent formula of torsion tensor is given by vectors $X = X^{\hat{j}} \mathbf{e}_{\hat{j}}$ and $Y = Y^{\hat{k}} \mathbf{e}_{\hat{k}}$ with $\mathbf{e}_{\hat{i}} := D_{\hat{i}} f$ of

$$\boxed{X^{\hat{j}} Y^{\hat{k}} T^{\hat{i}}{}_{\hat{j}\hat{k}} \mathbf{e}_{\hat{i}} = D_X Y - D_Y X - [X, Y] := T(X, Y) \,.} \tag{326}$$

In terms of the connections and structure constants, (325) can be written as

$$\begin{cases} \text{Holonomic:} & \boxed{T^i{}_{jk} = \Gamma^i{}_{kj} - \Gamma^i{}_{jk} \,,} & (327\text{a}) \\ \text{Anholonomic:} & \boxed{T^{\hat{i}}{}_{\hat{j}\hat{k}} = \omega^{\hat{i}}{}_{\hat{k}\hat{j}} - \omega^{\hat{i}}{}_{\hat{j}\hat{k}} - c^{\hat{i}}{}_{\hat{j}\hat{k}} \,.} & (327\text{b}) \end{cases}$$

Here we note that a vector X act on a function f and a avector Y are respectively given by

$$\boxed{X(f) = X^{\hat{j}} \mathbf{e}_{\hat{j}}(f)} \quad \text{and} \quad \boxed{X(Y) = X^{\hat{j}} \mathbf{e}_{\hat{j}}(Y^{\hat{k}}) \mathbf{e}_{\hat{k}} + X^{\hat{j}} Y^{\hat{k}} \mathbf{e}_{\hat{j}} \mathbf{e}_{\hat{k}} \,.} \tag{328}$$

In addition, the transformation formula for torsion tensor should be

$$T^{\hat{i}}{}_{\hat{j}\hat{k}} = a^{\hat{i}}{}_i a_{\hat{j}}{}^j a_{\hat{k}}{}^k T^i{}_{jk} \,. \tag{329}$$

Non-fixed frame and gauge transformation For general space $\mathcal{M}^n$, the vector $\boldsymbol{V} = V^a \boldsymbol{E}_a$ $(a = 1, \ldots, n)$ under local coordinate X^a can be spanned by a *non-fixed holonomic* frame $\boldsymbol{E}_a := \frac{\partial}{\partial X^a}$ with $d\boldsymbol{E}_a \neq 0$. The vector $\boldsymbol{E}_a$ can be spanned by another set of *anholonomic* frame $\mathbf{e}_{\hat{b}}$ given by a $GL(n, \mathbb{R})$ transformation

$$\boxed{\boldsymbol{E}_a = A^{\hat{b}}{}_a \mathbf{e}_{\hat{b}} \quad \text{with} \quad A^{\hat{b}}{}_a \in GL(n, \mathbb{R}) \,,} \tag{330}$$

and we also have coframe

$$\boxed{dX^a = A_{\hat{b}}{}^a \vartheta^{\hat{b}} \,,} \tag{331}$$

where we have defined the inverse

$$\boxed{A_{\hat{b}}{}^a := (A^{-1})^{\hat{b}}{}_a \,.} \tag{332}$$

Therefore,

$$d\boldsymbol{V} = dV^a \boldsymbol{E}_a + V^a d\boldsymbol{E}_a \tag{333}$$

and we have to introduce the *connection form* $\Gamma^b{}_a$ and $\omega^{\hat{b}}{}_{\hat{a}}$ for the frame $\boldsymbol{E}_a$ and $\mathbf{e}_{\hat{a}}$ respectively, which gives

$$\begin{cases} d\boldsymbol{E}_a = \Gamma^b{}_a \boldsymbol{E}_b \,, & (334a) \\ d\mathbf{e}_{\hat{a}} = \omega^{\hat{b}}{}_{\hat{a}} \mathbf{e}_{\hat{b}} \,. & (334b) \end{cases}$$

The differential of $\boldsymbol{V}$ can also be expressed as

$$d\boldsymbol{V} = dV^a \boldsymbol{E}_a + V^a \Gamma^b{}_a \boldsymbol{E}_b = \big(dV^a + V^b \Gamma^a{}_b\big) \boldsymbol{E}_a := (\bar{D}\boldsymbol{V})^a \boldsymbol{E}_a \,. \tag{335}$$

Remark. We note that there is *no* normal space $\mathcal{M}^{n\perp}$ of $\mathcal{M}^n$, therefore, we obtain

$$d\boldsymbol{V} = (d\boldsymbol{V})^\top = \bar{D}\boldsymbol{V} \,, \tag{336}$$

where the connection $\bar{D}$ is defined with respect to basis $\boldsymbol{E}_a$. As a result, the differetial operator d also represents the covariant derivative $\bar{D}$ on general space $\mathcal{M}^n$.

On the other hand, the differetial

$$d\boldsymbol{E}_a = dA^{\hat{b}}{}_a \mathbf{e}_{\hat{b}} + A^{\hat{b}}{}_a d\mathbf{e}_{\hat{b}} = dA^{\hat{b}}{}_a \mathbf{e}_{\hat{b}} + A^{\hat{b}}{}_a \omega^{\hat{c}}{}_{\hat{b}} \mathbf{e}_{\hat{c}} = \big(dA^{\hat{b}}{}_a + A^{\hat{c}}{}_a \omega^{\hat{b}}{}_{\hat{c}}\big) \mathbf{e}_{\hat{b}} \,. \tag{337}$$

From (330) and (334a), it implies the relation between two connection forms

$$\Gamma^c{}_a \boldsymbol{E}_c = \Gamma^c{}_a A^{\hat{b}}{}_c \mathbf{e}_{\hat{b}} = \big(dA^b{}_a + A^{\hat{c}}{}_a \omega^{\hat{b}}{}_{\hat{c}}\big) \mathbf{e}_{\hat{b}} \quad \Longrightarrow \quad \boxed{\Gamma^c{}_a = A_{\hat{b}}{}^c \big(dA^{\hat{b}}{}_a + \omega^{\hat{b}}{}_{\hat{c}} A^{\hat{c}}{}_a\big) \,,} \tag{338}$$

which is a *frame transformation* or $GL(n, \mathbb{R})$ *gauge transformation* of the connection form.

Remark. We note that the relation (338) comes from the *frame transformation* or $GL(n, \mathbb{R})$ *gauge transformation*, rather than *metric compatibility*. This relation is sometimes called *vielbein postulate*. However, the equation is still valid even if the frame $\mathbf{e}_{\hat{a}}$ is not orthonormal or the nonmetricity $Q_{abc} = -\nabla_a g_{bc}$ is not vanished. In such case, the connection $\omega^{\hat{b}}{}_{\hat{a}}$ contains the symmetric or trace part, *i.e.*, $\omega_{(\hat{a}\hat{b})} \neq 0$ or $\omega^{\hat{a}}{}_{\hat{a}} \neq 0$. So it is improper to call the relation postulate. People always *implicitly* define a *total* connection $\boldsymbol{D}(\Gamma, \omega)$ of tensor with respect to both the holonomic and anholonomic basis of $\boldsymbol{E}_a$, dX^a, $\mathbf{e}_{\hat{a}}$ and $\vartheta^{\hat{a}}$. By giving a tensor $A := A^{\hat{b}}{}_a \mathbf{e}_{\hat{b}} \otimes dX^a$, the connection $\boldsymbol{D}$ act on A is

$$\begin{aligned} \boldsymbol{D}A &= \boldsymbol{D}\big(A^{\hat{b}}{}_a \mathbf{e}_{\hat{b}} \otimes dX^a\big) \\ &= \big(dA^{\hat{b}}{}_{\hat{a}}\big) \mathbf{e}_{\hat{b}} \otimes dX^{\hat{a}} + A^{\hat{b}}{}_a \big(\boldsymbol{D}\mathbf{e}_{\hat{b}}\big) \otimes dX^a + A^{\hat{b}}{}_a \mathbf{e}_{\hat{b}} \otimes \big(\boldsymbol{D} dX^a\big) \\ &= \big(dA^{\hat{b}}{}_a\big) \mathbf{e}_{\hat{b}} \otimes dX^a + A^{\hat{b}}{}_a \big(\omega^{\hat{c}}{}_{\hat{b}} \mathbf{e}_{\hat{c}}\big) \otimes dX^a + A^{\hat{b}}{}_a \mathbf{e}_{\hat{b}} \otimes \big(-\Gamma^a{}_c dX^c\big) \\ &= \big(dA^{\hat{b}}{}_a + A^{\hat{c}}{}_a \omega^{\hat{b}}{}_{\hat{c}} - A^{\hat{b}}{}_c \Gamma^c{}_a\big) \mathbf{e}_{\hat{b}} \otimes dX^a = 0 \end{aligned} \tag{339}$$

due to (338), which is *independent* of the *metric compatibility*. As a result, the component gives the vielbein postulate

$$(\boldsymbol{D}A)^{\hat{b}}{}_a = dA^{\hat{b}}{}_a + A^{\hat{c}}{}_a \omega^{\hat{b}}{}_{\hat{c}} - A^{\hat{b}}{}_c \Gamma^c{}_a = 0 \tag{340}$$

or

$$\boldsymbol{\nabla}_d A^{\hat{b}}{}_a = \partial_d A^{\hat{b}}{}_a + A^{\hat{c}}{}_a \omega^{\hat{b}}{}_{\hat{c}d} - A^{\hat{b}}{}_c \Gamma^c{}_{ad} = 0 \,. \tag{341}$$

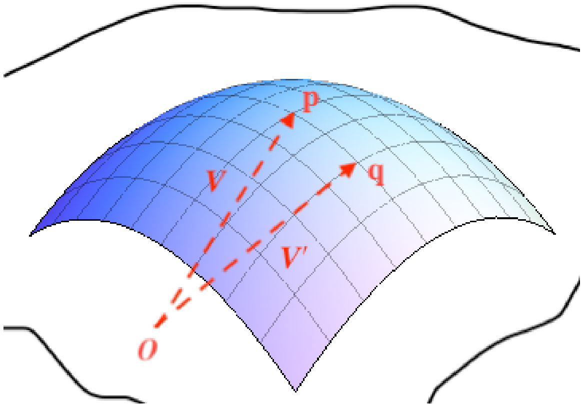


Figure 10: Two points on the hypersurface $\mathcal{M}^{n-1}$ in $\mathcal{M}^n$.

Now we will discuss the connection on the hypersurface $\mathcal{M}^{n-1}$ of $\overline{\mathcal{M}}^{\,n}$. Consider a point $\mathbf{p}$ on $\mathcal{M}^{n-1}$ is identified by a vector $\boldsymbol{V}$ in $\overline{\mathcal{M}}^{\,n}$. Simirlarly, a point $\mathbf{q}$ on $\mathcal{M}^{n-1}$ is represented by $\boldsymbol{V}'$ in $\overline{\mathcal{M}}^{\,n}$. Here we only focus on the connection on $\mathcal{M}^{n-1}$, we have restricted our case that $d\boldsymbol{V} = \mathbf{q}-\mathbf{p} = \boldsymbol{V}' - \boldsymbol{V}$ lays on the hypersurface $\mathcal{M}^{n-1}$ only. So $d\boldsymbol{V}$ can be expanded not only by frame $\boldsymbol{E}_a = \frac{\partial}{\partial X^a}$ $(a = 1, \ldots, n)$ on $\overline{\mathcal{M}}^{\,n}$ but also by frame $\partial_i = \frac{\partial}{\partial u^i}$ $(i = 1, \ldots, n-1)$ on $\mathcal{M}^{n-1}$.

Remark. We can consider the case of $n = 3$ and two infinitesimal closed points $\mathbf{q}$ and $\mathbf{p}$ with the spherical coordinate $X^a = (r, \theta, \phi)$ in $\overline{\mathcal{M}}^{\,3}$ and polar coordinate $u^i = (\rho, \varphi)$ in $\mathcal{M}^2$. Therefore $\boldsymbol{E}_a$ and ∂_i are non-fixed frames.

The differential dV^a can be given by

$$dV^a = \frac{\partial V^a}{\partial u} du + \frac{\partial V^a}{\partial v} dv = \partial_i V^a du^i \,, \tag{342}$$

then all the n-dimensional vectors can be expanded by $(n-1)$-dimensional ones, the resulting equation of (335) would be rewritten as

$$\begin{aligned} d\boldsymbol{V} &= (\bar{D}\boldsymbol{V})^a \boldsymbol{E}_a \\ &= \big(\partial_i V^a du^i + V^b \Gamma^a{}_{bc} dX^c\big) \boldsymbol{E}_a \\ &= \Big(\partial_i V^a du^i + V^b \Gamma^a{}_{bc} \underbrace{\frac{\partial X^c}{\partial u^i}}_{h^c{}_i} du^i\Big) \boldsymbol{E}_a \\ &= \big(\partial_i V^a + V^b \Gamma^a{}_{bc} h^c{}_i\big) du^i \boldsymbol{E}_a := V^a_i du^i \boldsymbol{E}_a \,. \end{aligned} \tag{343}$$

Here we define

$$\boxed{\boldsymbol{V}_i := V^a_i \boldsymbol{E}_a = \big(\partial_i V^a + V^b \Gamma^a{}_{bc} h^c{}_i\big) \boldsymbol{E}_a \,.} \tag{344}$$

Remark. The result of (66) in $\mathbb{E}^3$ can be reduced from (335) by

$$\begin{cases} \overline{\mathcal{M}}^{\,n} \longrightarrow \mathbb{E}^3 \\ V^a \longrightarrow p^a = x^a \,, \\ \boldsymbol{E}_a \longrightarrow \delta_a \,, \\ \Gamma^a{}_b \longrightarrow 0 \,, \\ (\bar{D}\boldsymbol{V})^a \longrightarrow dx^a \,. \end{cases} \tag{345}$$

Due to (334a) and (338), we have

$$0 = d\delta_a = dA^{\hat{b}}{}_a \mathbf{e}_{\hat{b}} + A^{\hat{b}}{}_a \omega^{\hat{c}}{}_{\hat{b}} \mathbf{e}_{\hat{c}} = (dA^{\hat{b}}{}_a + A^{\hat{c}}{}_a \omega^{\hat{b}}{}_{\hat{c}}) \mathbf{e}_{\hat{b}} \,. \tag{346}$$

As a consequence, the connection form $\omega^{\hat{b}}{}_{\hat{c}}$ is obtained by

$$\boxed{\omega^{\hat{b}}{}_{\hat{c}} = -A_{\hat{c}}{}^a \, dA^{\hat{b}}{}_a = +A^{\hat{b}}{}_a \, dA_{\hat{c}}{}^a \,.} \tag{347}$$

In addition, within (345), (343) becomes as

$$d\mathbf{p} = \big(\partial_i x^a + x^b \underbrace{\Gamma^a{}_{bc}}_{0} h^c{}_i\big) du^i \delta_a = \big(\partial_i x^a\big) du^i \delta_a := \partial_i \mathbf{p} du^i \,. \tag{348}$$

Therefore, (344) reduces to the derivative vector $\mathbf{p}_i := \partial_i \mathbf{p}$ on the hypersurface $\mathcal{M}$ of $\mathbb{E}^3$ is

$$\mathbf{p}_i = \partial_i \mathbf{p} = \partial_i x^a \delta_a = (\partial_i x, \partial_i y, \partial_i z) \,. \tag{349}$$

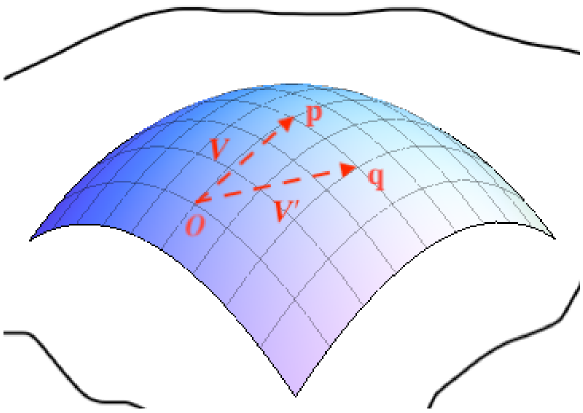


Figure 11: Two vectors on the hypersurface $\mathcal{M}^{n-1}$ in $\mathcal{M}^n$.

Induced connection However, if we move the reference point $\boldsymbol{o}$ on the hypersurface $\mathcal{M}^{n-1}$, the vector $d\boldsymbol{V}$ should be regarded as the difference between $\boldsymbol{V}'$ and $\boldsymbol{V}$ on $\mathcal{M}^{n-1}$ and is equivalent to $D\boldsymbol{V}$ with respect to the basis $\frac{\partial}{\partial u^i}$ as shown in Fig. 11. Then, we have

$$\begin{aligned} d\boldsymbol{V} = D\boldsymbol{V} = D\Big(V^k \frac{\partial}{\partial u^k}\Big) \\ = \big(\partial_i V^k + V^l \Gamma^k{}_{li}\big) du^i \frac{\partial}{\partial u^k} \,. \end{aligned} \tag{350}$$

By using chain rule to expand $\boldsymbol{E}_a$ in terms of $\frac{\partial}{\partial u^k}$

$$\boldsymbol{E}_a = \frac{\partial}{\partial X^a} = \underbrace{\frac{\partial u^k}{\partial X^a}}_{h_a{}^k} \frac{\partial}{\partial u^k} = h_a{}^k \frac{\partial}{\partial u^k} \tag{351}$$

and substituting the relation into (343), we have

$$d\boldsymbol{V} = \big(\partial_i V^a + V^b \Gamma^a{}_{bc} h^c{}_i\big) du^i \Big(h_a{}^k \frac{\partial}{\partial u^k}\Big)$$

$$= \Bigg(\partial_i\Big(V^j \underbrace{\frac{\partial X^a}{\partial u^j}}_{h^a{}_j}\Big) h_a{}^k + V^l \underbrace{\frac{\partial X^b}{\partial u^l}}_{h^b{}_l} \Gamma^a{}_{bc} h^c{}_i h_a{}^k\Bigg) du^i \frac{\partial}{\partial u^k}$$

$$= \Bigg((\partial_i V^j) \underbrace{h^a{}_j h_a{}^k}_{\delta^k_j} + V^l\Big((\partial_i h^a{}_l) h_a{}^k + h^b{}_l \Gamma^a{}_{bc} h^c{}_i h_a{}^k\Big)\Bigg) du^i \frac{\partial}{\partial u^k}$$

$$= \Bigg((\partial_i V^k) + V^l\Big((\partial_i h^a{}_l) h_a{}^k + h^b{}_l \Gamma^a{}_{bc} h^c{}_i h_a{}^k\Big)\Bigg) du^i \frac{\partial}{\partial u^k}\,. \tag{352}$$

Now we compare (350) and (352), the *induced connection* $\Gamma^k{}_{li}$ on hypersurface $\mathcal{M}^{n-1}$ can be obtained from the connection $\Gamma^a{}_{bc}$ on $\overline{\mathcal{M}}^{\,n}$ through the projection $h_a{}^k$

$$\boxed{\Gamma^k{}_{li} = (\partial_i h^a{}_l) h_a{}^k + h^b{}_l \Gamma^a{}_{bc} h^c{}_i h_a{}^k\,.} \tag{353}$$

We note that the discussion above can be generalized to the case for arbitrary frame (including fixed and non-fixed frame).

Curvature and torsion in subspace Now we would like to consider an n-dimensional space $\mathcal{M}$ embedded in a m-dimensional space $\overline{\mathcal{M}}$. We consider a so-called *Darboux frame* of $\mathcal{M}$. We label the components by indices $\hat{a}, \hat{b}, \hat{c} = \hat{1}, \ldots, \hat{m}$ on $\overline{\mathcal{M}}$ the indices $\hat{\imath}, \hat{\jmath}, \hat{k} = \hat{1}, \ldots, \hat{n}$ on $\mathcal{M}$, and the indices $p, q, r = \hat{n} + \hat{1}, \ldots, \hat{m}$ on the normal space $\mathcal{M}^{\perp}$ of $\mathcal{M}$ in the orthonormal frame. We define the geometric objects on $\overline{\mathcal{M}}$ specified by *barred* symbols. The frame $\bar{\mathbf{e}}_{\hat{a}}$ is extended by the vector $\mathbf{e}_{\hat{\imath}}$ and $\mathbf{e}_{\hat{p}}$

$$\bar{\mathbf{e}}_{\hat{a}} = \delta^{\hat{\imath}}_{\hat{a}}\, \mathbf{e}_{\hat{\imath}} + \delta^{\hat{p}}_{\hat{a}}\, \mathbf{e}_{\hat{p}} \quad \longrightarrow \quad \bar{e} := (\bar{\mathbf{e}}_{\hat{a}}) = \begin{pmatrix} \mathbf{e}_{\hat{\imath}} \\ \mathbf{e}_{\hat{p}} \end{pmatrix}. \tag{354}$$

Similarly, we have

$$\bar{\vartheta}^{\hat{a}} = \delta^{\hat{a}}_{\hat{\imath}}\, \vartheta^{\hat{\imath}} + \delta^{\hat{a}}_{\hat{p}}\, \vartheta^{\hat{p}} \quad \longrightarrow \quad \bar{\boldsymbol{\vartheta}} := (\bar{\vartheta}^{\hat{a}}) = \begin{pmatrix} \vartheta^{\hat{\imath}} & \vartheta^{\hat{p}} \end{pmatrix}. \tag{355}$$

Therefore, the components of $\{\bar{\mathbf{e}}_{\hat{a}}\}$ and $\{\bar{\vartheta}^{\hat{a}}\}$ are $\bar{\mathbf{e}}_{\hat{\imath}} = \mathbf{e}_{\hat{\imath}}$, $\bar{\mathbf{e}}_{\hat{p}} = \mathbf{e}_{\hat{p}}$, $\bar{\vartheta}^{\hat{\imath}} = \vartheta^{\hat{\imath}}$ and $\bar{\vartheta}^{\hat{p}} = \vartheta^{\hat{p}}$. The metric is defined by the inner product of two vectors, we have the following relations

$$\bar{g}(\bar{\mathbf{e}}_{\hat{a}}, \bar{\mathbf{e}}_{\hat{b}}) = \delta_{\hat{a}\hat{b}}\,, \quad g(\mathbf{e}_{\hat{\imath}}, \mathbf{e}_{\hat{\jmath}}) = \delta_{\hat{\imath}\hat{\jmath}}\,, \quad g^{\perp}(\mathbf{e}_{\hat{p}}, \mathbf{e}_{\hat{q}}) = \delta_{\hat{p}\hat{q}}\,, \quad \bar{g}(\bar{\mathbf{e}}_{\hat{\imath}}, \bar{\mathbf{e}}_{\hat{p}}) = 0\,, \tag{356}$$

where $\bar{g}$, g and $g^{\perp}$ are metrics of the manifolds $\overline{\mathcal{M}}$, $\mathcal{M}$ and $\mathcal{M}^{\perp}$ respectively. Due to the metric compatible condition, we have $\bar{\omega}^{\hat{b}}{}_{\hat{a}} = -\bar{\omega}^{\hat{a}}{}_{\hat{b}}$ and

$$\bar{\omega}^{\hat{b}}{}_{\hat{a}} = \delta^{\hat{b}}_{\hat{\jmath}}\delta^{\hat{\imath}}_{\hat{a}}\omega^{\hat{\jmath}}{}_{\hat{\imath}} + \delta^{\hat{b}}_{\hat{q}}\delta^{\hat{\imath}}_{\hat{a}}\omega^{\hat{q}}{}_{\hat{\imath}} + \delta^{\hat{b}}_{\hat{\jmath}}\delta^{\hat{p}}_{\hat{a}}\omega^{\hat{\jmath}}{}_{\hat{p}} + \delta^{\hat{b}}_{\hat{q}}\delta^{\hat{p}}_{\hat{a}}\omega^{\hat{q}}{}_{\hat{p}}\,, \tag{357}$$

which can also be recognized by the matrix as

$$\boldsymbol{\omega} := (\bar{\omega}^{\hat{b}}{}_{\hat{a}}) = \begin{pmatrix} \omega^{\hat{\jmath}}{}_{\hat{\imath}} & \omega^{\hat{q}}{}_{\hat{\imath}} \\ \omega^{\hat{\jmath}}{}_{\hat{p}} & \omega^{\hat{q}}{}_{\hat{p}} \end{pmatrix}. \tag{358}$$

Now we will discuss the dynamics on subspace $\mathcal{M}$ *only*. The differential of frame on $\mathcal{M}$ are given by

$$\begin{cases} \mathbf{d}_{\nabla}\mathbf{p} = \bar{\vartheta}^{\hat{a}}\bar{\mathbf{e}}_{\hat{a}} \quad \text{with} \quad \vartheta^{\hat{p}} = 0\,, & (359\text{a}) \\ \mathbf{d}_{\nabla}\bar{\mathbf{e}}_{\hat{a}} = \bar{\omega}^{\hat{b}}{}_{\hat{a}}\bar{\mathbf{e}}_{\hat{b}}\,. & (359\text{b}) \end{cases}$$

or

$$\begin{cases} \mathbf{d}_\nabla \mathbf{p} = \vartheta^{\hat{i}} \mathbf{e}_{\hat{i}} \,, \\ \mathbf{d}_\nabla \mathbf{e}_{\hat{i}} = \omega^{\hat{j}}{}_{\hat{i}} \mathbf{e}_{\hat{j}} + \omega^{\hat{q}}{}_{\hat{i}} \mathbf{e}_{\hat{q}} \,, \\ \mathbf{d}_\nabla \mathbf{e}_{\hat{p}} = \omega^{\hat{j}}{}_{\hat{p}} \mathbf{e}_{\hat{j}} + \omega^{\hat{q}}{}_{\hat{p}} \mathbf{e}_{\hat{q}} \,. \end{cases} \longrightarrow \begin{pmatrix} \mathbf{d}_\nabla \mathbf{p} \\ \mathbf{d}_\nabla \mathbf{e}_{\hat{j}} \\ \mathbf{d}_\nabla \mathbf{e}_{\hat{p}} \end{pmatrix} = \begin{pmatrix} \vartheta^{\hat{i}} & 0 \\ \omega^{\hat{j}}{}_{\hat{i}} & \omega^{\hat{q}}{}_{\hat{i}} \\ \omega^{\hat{j}}{}_{\hat{p}} & \omega^{\hat{q}}{}_{\hat{p}} \end{pmatrix} \begin{pmatrix} \mathbf{e}_{\hat{i}} \\ \mathbf{e}_{\hat{q}} \end{pmatrix} . \tag{360a, 360b, 360c}$$

In addition

$$\vartheta^{\hat{p}} = 0 \qquad \Longrightarrow \qquad 0 = \mathbf{D}\vartheta^{\hat{p}} = \mathbf{d}\underbrace{\vartheta^{\hat{p}}}_{0} + \omega^{\hat{p}}{}_{\hat{i}} \wedge \vartheta^{\hat{i}} + \omega^{\hat{p}}{}_{\hat{q}} \wedge \underbrace{\vartheta^{\hat{q}}}_{0} = \overline{\mathcal{T}}^{\hat{p}} \,, \tag{361}$$

Thus, we have

$$\boxed{\omega^{\hat{p}}{}_{\hat{i}} \wedge \vartheta^{\hat{i}} = 0 \,.} \tag{362}$$

Applying the Cartan's lemma, we obtain the connection form

$$\boxed{\omega^{\hat{p}}{}_{\hat{i}} = -\omega^{\hat{i}}{}_{\hat{p}} = h^{\hat{p}}{}_{\hat{i}\hat{j}} \vartheta^{\hat{j}} \,,} \tag{363}$$

or

$$\boxed{\omega_{\hat{p}\hat{i}} = h_{\hat{p}\hat{i}\hat{j}} \vartheta^{\hat{j}} \quad \text{and} \quad \omega_{\hat{i}\hat{p}} = -\omega_{\hat{p}\hat{i}} = -h_{\hat{p}\hat{i}\hat{j}} \vartheta^{\hat{j}} = h_{\hat{i}\hat{p}\hat{j}} \vartheta^{\hat{j}} \quad \text{(in \textit{pseudo-Riemannian} geometry)} \,.} \tag{364}$$

Now we would like to calculate the differential of structure equations. The covarint exterior differentiation of first structure equation is obtained by

$$\begin{aligned} \mathbf{d}^2_\nabla \mathbf{p} &= \mathbf{d}_\nabla (\bar{\vartheta}^{\hat{a}} \bar{\mathbf{e}}_{\hat{a}}) = \mathbf{d}_\nabla (\vartheta^{\hat{i}} \mathbf{e}_{\hat{i}} + \vartheta^{\hat{p}} \mathbf{e}_{\hat{p}}) \\ &= \mathbf{d}\vartheta^{\hat{i}} \mathbf{e}_{\hat{i}} - \vartheta^{\hat{i}} \wedge \mathbf{d}_\nabla \mathbf{e}_{\hat{i}} + \mathbf{d}\vartheta^{\hat{p}} \mathbf{e}_{\hat{p}} - \vartheta^{\hat{p}} \wedge \mathbf{d}_\nabla \mathbf{e}_{\hat{p}} \\ &= \mathbf{d}\vartheta^{\hat{i}} \mathbf{e}_{\hat{i}} - \vartheta^{\hat{i}} \wedge \left(\omega^{\hat{j}}{}_{\hat{i}} \mathbf{e}_{\hat{j}} + \omega^{\hat{p}}{}_{\hat{i}} \mathbf{e}_{\hat{p}} \right) + \mathbf{d}\vartheta^{\hat{p}} \mathbf{e}_{\hat{p}} - \vartheta^{\hat{p}} \wedge \left(\omega^{\hat{i}}{}_{p} \mathbf{e}_{\hat{i}} + \omega^{\hat{q}}{}_{\hat{p}} \mathbf{e}_{\hat{q}} \right) \\ &= \mathbf{d}\vartheta^{i} \mathbf{e}_{i} + \omega^{\hat{j}}{}_{\hat{i}} \wedge \vartheta^{\hat{i}} \mathbf{e}_{\hat{j}} + \omega^{\hat{p}}{}_{\hat{i}} \wedge \vartheta^{\hat{i}} \mathbf{e}_{\hat{p}} + \mathbf{d}\vartheta^{\hat{p}} \mathbf{e}_{\hat{p}} + \omega^{\hat{i}}{}_{\hat{p}} \wedge \vartheta^{\hat{p}} \mathbf{e}_{\hat{i}} + \omega^{\hat{q}}{}_{\hat{p}} \wedge \vartheta^{\hat{p}} \mathbf{e}_{\hat{q}} \\ &= \big(\underbrace{\mathbf{d}\vartheta^{\hat{i}} + \omega^{\hat{i}}{}_{\hat{j}} \wedge \vartheta^{\hat{i}}}_{\mathcal{T}^{\hat{i}}} + \omega^{\hat{i}}{}_{\hat{p}} \wedge \vartheta^{\hat{p}} \big) \mathbf{e}_{\hat{i}} + \big(\underbrace{\mathbf{d}\vartheta^{\hat{p}} + \omega^{\hat{p}}{}_{\hat{q}} \wedge \vartheta^{\hat{q}}}_{\mathcal{T}^{\hat{p}}} + \omega^{\hat{p}}{}_{\hat{i}} \wedge \vartheta^{\hat{i}} \big) \mathbf{e}_{\hat{p}} \\ &= \overline{\mathcal{T}}^{\hat{a}} \bar{\mathbf{e}}_{\hat{a}} = \overline{\mathcal{T}}^{\hat{i}} \mathbf{e}_{\hat{i}} + \overline{\mathcal{T}}^{\hat{p}} \mathbf{e}_{\hat{p}} \,. \end{aligned} \tag{365}$$

By using (361), we obtain

$$\mathbf{d}^2_\nabla \mathbf{p} = \overline{\mathcal{T}}^{\hat{i}} \mathbf{e}_{\hat{i}} = \big(\mathcal{T}^{\hat{i}} + \omega^{\hat{p}}{}_{\hat{q}} \wedge \underbrace{\vartheta^{\hat{q}}}_{0} \big) \mathbf{e}_{\hat{i}} = \mathcal{T}^{\hat{i}} \mathbf{e}_{\hat{i}} \,, \tag{366}$$

which leads to the equation

$$\boxed{\overline{\mathcal{T}}^{\hat{i}} = \mathcal{T}^{\hat{i}} \,.} \tag{367}$$

Remark. For case of $\mathcal{M}$ is embedded in $\overline{\mathcal{M}}$, we have consquence of

$$\boxed{\overline{\mathcal{T}}^{\hat{i}} = \mathcal{T}^{\hat{i}} \,.} \tag{368}$$

It means that there is *no extrinsic torsion* contribution in the equation of torsion in embedding structure of geometry (*cf.* Gauss equation (372a)).

The covarint exterior differentiation of second structure equation is

$$\mathbf{d}^2_\nabla \bar{\mathbf{e}}_{\hat{a}} = \overline{\mathcal{R}}^{\hat{a}}{}_{\hat{a}} \bar{\mathbf{e}}_{\hat{a}} = \overline{\mathcal{R}}^{\hat{j}}{}_{\hat{a}} \mathbf{e}_{\hat{j}} + \overline{\mathcal{R}}^{\hat{p}}{}_{\hat{a}} \mathbf{e}_{\hat{p}} \,, \tag{369}$$

which can be calculated separately by $\mathbf{d}^2_\nabla \mathbf{e}_{\hat{i}}$ and $\mathbf{d}^2_\nabla \mathbf{e}_{\hat{p}}$. They are shown by

$$\begin{aligned}
\mathbf{d}^2_\nabla \mathbf{e}_{\hat{i}} &= \mathbf{d}\omega^{\hat{j}}{}_{\hat{i}} \mathbf{e}_{\hat{j}} - \omega^{\hat{j}}{}_{i} \wedge \mathbf{d}_\nabla \mathbf{e}_{\hat{j}} + \mathbf{d}\omega^{\hat{p}}{}_{\hat{i}} \mathbf{e}_{\hat{p}} - \omega^{\hat{p}}{}_{\hat{i}} \wedge \mathbf{d}_\nabla \mathbf{e}_{\hat{p}} \\
&= \mathbf{d}\omega^{\hat{j}}{}_{\hat{i}} \mathbf{e}_{\hat{j}} - \omega^{\hat{j}}{}_{\hat{i}} \wedge \left(\omega^{\hat{k}}{}_{\hat{j}} \mathbf{e}_{\hat{k}} + \omega^{\hat{p}}{}_{\hat{j}} \mathbf{e}_{\hat{p}}\right) + \mathbf{d}\omega^{\hat{p}}{}_{\hat{i}} \mathbf{e}_{\hat{p}} - \omega^{\hat{p}}{}_{\hat{i}} \wedge \left(\omega^{\hat{j}}{}_{\hat{p}} \mathbf{e}_{\hat{j}} + \omega^{\hat{q}}{}_{\hat{p}} \mathbf{e}_{\hat{q}}\right) \\
&= \mathbf{d}\omega^{\hat{j}}{}_{\hat{i}} \mathbf{e}_{\hat{j}} + \omega^{\hat{k}}{}_{\hat{j}} \wedge \omega^{\hat{j}}{}_{\hat{i}} \mathbf{e}_{\hat{k}} + \omega^{\hat{p}}{}_{\hat{j}} \wedge \omega^{\hat{j}}{}_{\hat{i}} \mathbf{e}_{\hat{p}} + \mathbf{d}\omega^{\hat{p}}{}_{\hat{i}} \mathbf{e}_{\hat{p}} + \omega^{\hat{j}}{}_{\hat{p}} \wedge \omega^{\hat{p}}{}_{\hat{i}} \mathbf{e}_{\hat{j}} + \omega^{\hat{q}}{}_{\hat{p}} \wedge \omega^{\hat{p}}{}_{\hat{i}} \mathbf{e}_{\hat{q}} \\
&= \Big(\underbrace{\mathbf{d}\omega^{\hat{j}}{}_{\hat{i}} + \omega^{\hat{j}}{}_{\hat{k}} \wedge \omega^{\hat{k}}{}_{\hat{i}}}_{\mathcal{R}^{\hat{j}}{}_{\hat{i}}} + \omega^{\hat{j}}{}_{\hat{p}} \wedge \omega^{\hat{p}}{}_{\hat{i}}\Big) \mathbf{e}_{\hat{j}} + \left(\mathbf{d}\omega^{\hat{p}}{}_{\hat{i}} + \omega^{\hat{p}}{}_{\hat{j}} \wedge \omega^{\hat{j}}{}_{\hat{i}} + \omega^{\hat{p}}{}_{\hat{q}} \wedge \omega^{\hat{q}}{}_{\hat{i}}\right) \mathbf{e}_{\hat{p}} \\
&= \overline{\mathcal{R}}^{\hat{j}}{}_{\hat{i}} \mathbf{e}_{\hat{j}} + \overline{\mathcal{R}}^{\hat{p}}{}_{\hat{i}} \mathbf{e}_{\hat{p}} = \overline{\mathcal{R}}^{\hat{a}}{}_{\hat{i}} \bar{\mathbf{e}}_{\hat{a}}
\end{aligned} \tag{370}$$

and

$$\begin{aligned}
\mathbf{d}^2_\nabla \mathbf{e}_{\hat{p}} &= \mathbf{d}\omega^{\hat{i}}{}_{\hat{p}} \mathbf{e}_{\hat{i}} - \omega^{\hat{i}}{}_{\hat{p}} \wedge \mathbf{d}_\nabla \mathbf{e}_{\hat{i}} + \mathbf{d}\omega^{\hat{q}}{}_{\hat{p}} \mathbf{e}_{\hat{q}} - \omega^{\hat{q}}{}_{\hat{p}} \wedge \mathbf{d}_\nabla \mathbf{e}_{\hat{q}} \\
&= \mathbf{d}\omega^{\hat{i}}{}_{\hat{p}} \mathbf{e}_{\hat{i}} - \omega^{\hat{i}}{}_{\hat{p}} \wedge \left(\omega^{\hat{j}}{}_{\hat{i}} \mathbf{e}_{\hat{j}} + \omega^{\hat{q}}{}_{\hat{i}} \mathbf{e}_{\hat{q}}\right) + \mathbf{d}\omega^{\hat{q}}{}_{\hat{p}} \mathbf{e}_{\hat{q}} - \omega^{\hat{q}}{}_{\hat{p}} \wedge \left(\omega^{\hat{i}}{}_{\hat{q}} \mathbf{e}_{\hat{i}} + \omega^{\hat{r}}{}_{\hat{q}} \mathbf{e}_{\hat{r}}\right) \\
&= \mathbf{d}\omega^{\hat{i}}{}_{\hat{p}} \mathbf{e}_{\hat{i}} + \omega^{\hat{j}}{}_{\hat{i}} \wedge \omega^{\hat{i}}{}_{\hat{p}} \mathbf{e}_{\hat{j}} + \omega^{\hat{q}}{}_{\hat{i}} \wedge \omega^{\hat{i}}{}_{\hat{p}} \mathbf{e}_{\hat{q}} + \mathbf{d}\omega^{\hat{q}}{}_{\hat{p}} \mathbf{e}_{\hat{q}} + \omega^{\hat{i}}{}_{\hat{q}} \wedge \omega^{\hat{q}}{}_{\hat{p}} \mathbf{e}_{\hat{i}} + \omega^{\hat{r}}{}_{\hat{q}} \wedge \omega^{\hat{q}}{}_{\hat{p}} \mathbf{e}_{\hat{r}} \\
&= \left(\mathbf{d}\omega^{\hat{i}}{}_{\hat{p}} + \omega^{\hat{i}}{}_{\hat{j}} \wedge \omega^{\hat{j}}{}_{\hat{p}} + \omega^{\hat{i}}{}_{\hat{q}} \wedge \omega^{\hat{q}}{}_{\hat{p}}\right) \mathbf{e}_{\hat{i}} + \Big(\underbrace{\mathbf{d}\omega^{\hat{q}}{}_{\hat{p}} + \omega^{\hat{q}}{}_{\hat{r}} \wedge \omega^{\hat{r}}{}_{\hat{p}}}_{\mathcal{R}^{\hat{q}}{}_{\hat{p}}} + \omega^{\hat{q}}{}_{\hat{i}} \wedge \omega^{\hat{i}}{}_{\hat{p}}\Big) \mathbf{e}_{\hat{q}} \\
&= \overline{\mathcal{R}}^{\hat{i}}{}_{\hat{p}} \mathbf{e}_{\hat{i}} + \overline{\mathcal{R}}^{\hat{q}}{}_{\hat{p}} \mathbf{e}_{\hat{q}} = \overline{\mathcal{R}}^{\hat{a}}{}_{\hat{p}} \bar{\mathbf{e}}_{\hat{a}} \,,
\end{aligned} \tag{371}$$

respectively. According to the results above, we have the following equations:

$$\begin{cases}
\text{Gauss equation:} \quad \overline{\mathcal{R}}^{\hat{j}}{}_{\hat{i}} = \mathcal{R}^{\hat{j}}{}_{\hat{i}} + \omega^{\hat{j}}{}_{\hat{p}} \wedge \omega^{\hat{p}}{}_{\hat{i}} \,, & (372a) \\
\text{Codazzi equation:} \quad \overline{\mathcal{R}}^{\hat{p}}{}_{\hat{i}} = \mathbf{d}\omega^{\hat{p}}{}_{\hat{i}} + \omega^{\hat{p}}{}_{\hat{j}} \wedge \omega^{\hat{j}}{}_{\hat{i}} + \omega^{\hat{p}}{}_{\hat{q}} \wedge \omega^{\hat{q}}{}_{\hat{i}} \,, & (372b) \\
\text{Ricci equation:} \quad \overline{\mathcal{R}}^{\hat{q}}{}_{\hat{p}} = \mathcal{R}^{\hat{q}}{}_{\hat{p}} + \omega^{\hat{q}}{}_{\hat{i}} \wedge \omega^{\hat{i}}{}_{\hat{p}} \,. & (372c)
\end{cases}$$

Subspace of $\mathbb{E}^m$ We consider that a space $\mathcal{M}$ is embedded in the flat space $\mathbb{E}^m$. We can chose the cartesian coordinate for $\mathbb{E}^m$, every component of the orthonormal frame $\{\bar{\mathbf{e}}_{\hat{a}}\}$ is related to the fixed cartesian frame by

$$\begin{cases}
\bar{\mathbf{e}}_{\hat{i}} = a_{\hat{i}}{}^{j} \delta_j = a_{\hat{i}}{}^{j} \dfrac{\partial}{\partial x^j} \,, \quad \text{and} \quad \bar{\vartheta}^{\hat{i}} = a^{\hat{i}}{}_{j} dx^j \quad (i = 1, 2, \ldots, n) \,, \\
\bar{\mathbf{e}}_{\hat{p}} = a_{\hat{p}}{}^{q} \delta_q = a_{\hat{p}}{}^{q} \dfrac{\partial}{\partial x^q} \,, \quad \text{and} \quad \bar{\vartheta}^{\hat{p}} = a^{\hat{p}}{}_{q} dx^q \quad (p = n+1, \ldots, m) \,,
\end{cases} \tag{373}$$

where $a_{\hat{i}}{}^{j}, a^{\hat{i}}{}_{j} \in SO(m, \mathbb{R})$. Therefore, the differential of the frame on subspace $\mathcal{M}$ with $\bar{\vartheta}^{\hat{p}} = 0$ is given by

$$\begin{cases}
\mathbf{d}_\nabla \mathbf{p} = \vartheta^{\hat{i}} \otimes \mathbf{e}_{\hat{i}} = dx^i \otimes \delta_i \,, & (374a) \\
\mathbf{d}_\nabla \bar{\mathbf{e}}_{\hat{a}} = \bar{\omega}^{\hat{b}}{}_{\hat{a}} \otimes \bar{\mathbf{e}}_{\hat{b}} = \bar{\Gamma}^{b}{}_{a} \otimes \delta_b \,, & (374b)
\end{cases}$$

and we can show that the torsion and curvature are vanished by using (374) in terms of cartesian frame, which gives the following equations for frame with torsion-free and curvature-free on $\mathbb{E}^m$

$$\begin{cases}
\mathbf{d}^2_\nabla \mathbf{p} = \overline{\mathcal{T}}^{\hat{i}} \mathbf{e}_{\hat{i}} = 0 \,, & (375a) \\
\mathbf{d}^2_\nabla \mathbf{e}_{\hat{i}} = \overline{\mathcal{R}}^{\hat{a}}{}_{\hat{i}} \bar{\mathbf{e}}_{\hat{a}} = 0 \,, & (375b) \\
\mathbf{d}^2_\nabla \mathbf{e}_{\hat{p}} = \overline{\mathcal{R}}^{\hat{a}}{}_{\hat{p}} \bar{\mathbf{e}}_{\hat{a}} = 0 \,. & (375c)
\end{cases}$$

The equations (367) and (372) turn out to be

$$\begin{cases} \text{Torsion-free:} & \mathcal{T}^{\hat{i}} = \mathbf{d}\vartheta^{\hat{i}} + \omega^{\hat{i}}{}_{\hat{j}} \wedge \vartheta^{\hat{j}} = 0\,, & \text{(376a)} \\ \text{Gauss equation:} & \mathcal{R}^{\hat{j}}{}_{\hat{i}} = -\omega^{\hat{j}}{}_{\hat{p}} \wedge \omega^{\hat{p}}{}_{\hat{i}} = \omega^{\hat{p}}{}_{\hat{j}} \wedge \omega^{\hat{p}}{}_{\hat{i}}\,, & \text{(376b)} \\ \text{Codazzi equation:} & 0 = \mathbf{d}\omega^{\hat{p}}{}_{\hat{i}} + \omega^{\hat{p}}{}_{\hat{j}} \wedge \omega^{\hat{j}}{}_{\hat{i}} + \omega^{p}{}_{\hat{q}} \wedge \omega^{\hat{q}}{}_{\hat{i}}\,, & \text{(376c)} \\ \text{Ricci equation:} & \mathcal{R}^{\hat{q}}{}_{\hat{p}} = -\omega^{\hat{q}}{}_{\hat{i}} \wedge \omega^{\hat{i}}{}_{\hat{p}} = \omega^{\hat{q}}{}_{\hat{i}} \wedge \omega^{\hat{p}}{}_{\hat{i}}\,, & \text{(376d)} \end{cases}$$

because all barred torsion and curvature 2-forms should be vanished in $\mathbb{E}^m$. According to (376a), we have torsion-free, it leads us to have Ricci rotation coefficients written as (316). From the consequence of (363), the Gauss equation (376b) is

$$\begin{aligned} \mathcal{R}^{\hat{j}}{}_{\hat{i}} &= \frac{1}{2} R^{\hat{j}}{}_{\hat{i}\hat{k}\hat{l}} \vartheta^{\hat{k}} \wedge \vartheta^{\hat{l}} \\ &= (h^{\hat{p}}{}_{\hat{j}\hat{k}} \vartheta^{\hat{k}}) \wedge (h^{\hat{p}}{}_{\hat{i}\hat{l}} \vartheta^{\hat{l}}) = \frac{1}{2}\big(h^{\hat{p}}{}_{\hat{j}\hat{k}} h^{\hat{p}}{}_{\hat{i}\hat{l}} - h^{\hat{p}}{}_{\hat{j}\hat{l}} h^{\hat{p}}{}_{\hat{i}\hat{k}}\big) \vartheta^{\hat{k}} \wedge \vartheta^{\hat{l}}\,, \end{aligned} \tag{377}$$

i.e.,

$$\boxed{R^{\hat{j}}{}_{\hat{i}\hat{k}\hat{l}} = h^{\hat{p}}{}_{\hat{j}\hat{k}} h^{\hat{p}}{}_{\hat{i}\hat{l}} - h^{\hat{p}}{}_{\hat{j}\hat{l}} h^{\hat{p}}{}_{\hat{i}\hat{k}}\,,} \tag{378}$$

or

$$\boxed{R^{\hat{j}}{}_{\hat{i}\hat{k}\hat{l}} = -h^{\hat{j}}{}_{\hat{p}\hat{k}} h^{\hat{p}}{}_{\hat{i}\hat{l}} + h^{\hat{j}}{}_{\hat{p}\hat{l}} h^{\hat{p}}{}_{\hat{i}\hat{k}} = h_{\hat{p}}{}^{\hat{j}}{}_{\hat{k}} h^{\hat{p}}{}_{\hat{i}\hat{l}} - h_{\hat{p}}{}^{\hat{j}}{}_{\hat{l}} h^{\hat{p}}{}_{\hat{i}\hat{k}} \quad (\text{in } \textit{pseudo-Riemannian} \text{ geometry})\,.} \tag{379}$$

The Codazzi equation (376c) becomes

$$\boxed{\begin{aligned} 0 &= \mathbf{d}(h^{\hat{p}}{}_{\hat{i}\hat{j}} \vartheta^{\hat{j}}) + (h^{\hat{p}}{}_{\hat{j}\hat{k}} \vartheta^{\hat{k}}) \wedge \omega^{\hat{j}}{}_{\hat{i}} + \omega^{\hat{p}}{}_{\hat{q}} \wedge (h^{\hat{q}}{}_{\hat{i}\hat{j}} \vartheta^{\hat{j}}) \\ &= dh^{\hat{p}}{}_{\hat{i}\hat{j}} \wedge \vartheta^{\hat{j}} + h^{\hat{p}}{}_{\hat{i}\hat{j}} \mathbf{d}\vartheta^{\hat{j}} + h^{\hat{p}}{}_{\hat{j}\hat{k}} \vartheta^{\hat{k}} \wedge \omega^{\hat{j}}{}_{\hat{i}} + h^{\hat{q}}{}_{\hat{i}\hat{j}} \omega^{\hat{p}}{}_{\hat{q}} \wedge \vartheta^{\hat{j}}\,. \end{aligned}} \tag{380}$$

The Ricci equation can be read as

$$\begin{aligned} \mathcal{R}^{\hat{q}}{}_{\hat{p}} &= \frac{1}{2} R^{\hat{q}}{}_{\hat{p}\hat{k}\hat{l}} \vartheta^{\hat{k}} \wedge \vartheta^{\hat{l}} \\ &= (h^{\hat{q}}{}_{\hat{i}\hat{k}} \vartheta^{\hat{k}}) \wedge (h^{\hat{p}}{}_{\hat{i}\hat{l}} \vartheta^{\hat{l}}) = \frac{1}{2}\big(h^{\hat{q}}{}_{\hat{i}\hat{k}} h^{\hat{p}}{}_{\hat{i}\hat{l}} - h^{\hat{q}}{}_{\hat{i}\hat{l}} h^{\hat{p}}{}_{\hat{i}\hat{k}}\big) \vartheta^{\hat{k}} \wedge \vartheta^{\hat{l}}\,, \end{aligned} \tag{381}$$

i.e.,

$$\boxed{R^{\hat{q}}{}_{\hat{p}\hat{k}\hat{l}} = h^{\hat{q}}{}_{\hat{i}\hat{k}} h^{\hat{p}}{}_{\hat{i}\hat{l}} - h^{\hat{q}}{}_{\hat{i}\hat{l}} h^{\hat{p}}{}_{\hat{i}\hat{k}}\,,} \tag{382}$$

or

$$\boxed{R^{\hat{q}}{}_{\hat{p}\hat{k}\hat{l}} = -h^{\hat{q}}{}_{\hat{i}\hat{k}} h^{\hat{i}}{}_{\hat{p}\hat{l}} + h^{\hat{q}}{}_{\hat{i}\hat{l}} h^{\hat{i}}{}_{\hat{p}\hat{k}} = h^{\hat{q}}{}_{\hat{i}\hat{k}} h_{\hat{p}}{}^{\hat{i}}{}_{\hat{l}} - h^{\hat{q}}{}_{\hat{i}\hat{l}} h_{\hat{p}}{}^{\hat{i}}{}_{\hat{k}} \quad (\text{in } \textit{pseudo-Riemannian} \text{ geometry})\,.} \tag{383}$$

Example (Hypersurface of $\mathbb{E}^3$). If we consider $\mathcal{M}$ and $\overline{\mathcal{M}}$ to be $\mathcal{M}^2$ and $\mathbb{E}^3$ and $\frac{\partial}{\partial x^3}$ is assume to be aligned to the normal vector $\mathbf{n}$ of $\mathcal{M}$, we have

$$\begin{cases} \bar{\mathbf{e}}_{\hat{i}} = a_{\hat{i}}{}^{j} \delta_j = a_{\hat{i}}{}^{j} \dfrac{\partial}{\partial x^j}\,, \quad \text{and} \quad \bar{\vartheta}^{i} = a^{\hat{i}}{}_{j} dx^j \quad (i,j = 1,2 \text{ and } x^1 = x, x^2 = y)\,, \\ \bar{\mathbf{e}}_{\hat{3}} = a_{\hat{3}}{}^{3} \delta_3 = a_{\hat{3}}{}^{3} \dfrac{\partial}{\partial x^3}\,, \quad \text{and} \quad \bar{\vartheta}^{\hat{3}} = a^{\hat{3}}{}_{3} dx^3 \quad (x^p = x^3 = z)\,. \end{cases} \tag{384a}$$

Due to the fixed condition $p = n + 1 = m = 3$, it is impossible to have $p \neq q$, which leads to the results for *hypersuface* with $\bar{\vartheta}^{\hat{3}} = 0$ of

$$\boxed{\mathcal{R}^{\hat{q}}{}_{\hat{p}} = 0 \quad \text{(for hypersuface)}\,,} \tag{385}$$

and

$$\boxed{\omega^{\hat{p}}{}_{\hat{q}} = 0 \quad \text{(for hypersuface)}\,.} \tag{386}$$

We can identify $h^{\hat{3}}{}_{\hat{i}\hat{j}}$ to be $b_{\hat{i}\hat{j}}$ which is the extrinsic curvature of $\mathcal{M}$. The corresponding component equations of (378) and (380) are

$$\begin{cases} R^{\hat{j}}{}_{\hat{i}\hat{k}\hat{l}} = b_{\hat{j}\hat{k}} b_{\hat{i}\hat{l}} - b_{\hat{j}\hat{l}} b_{\hat{i}\hat{k}}\,, & \text{(387a)} \\ \quad 0 = db_{\hat{i}\hat{j}} \wedge \vartheta^{\hat{j}} + b_{\hat{i}\hat{j}} \mathbf{d}\vartheta^{\hat{j}} + b_{\hat{j}\hat{k}} \omega^{\hat{j}}{}_{\hat{i}\hat{l}} \vartheta^{\hat{k}} \wedge \vartheta^{\hat{l}}\,. & \text{(387b)} \end{cases}$$

If we use the holonomic frame with coordinate $\{u^i\}$ on $\mathcal{M}$, we have

$$\vartheta^{\hat{i}} = a^{\hat{i}}{}_j dx^j = a^{\hat{i}}{}_j \frac{\partial x^j}{\partial u^k} du^k = du^i \qquad \Longrightarrow \qquad e^{\hat{i}}{}_k := a^{\hat{i}}{}_j \frac{\partial x^j}{\partial u^k} = \delta^{\hat{i}}{}_k \tag{388}$$

such that $\mathbf{d}\vartheta^{\hat{i}} = \mathbf{d}du^i = 0$ (or $c^{\hat{i}}{}_{\hat{j}\hat{k}} = 0$) and $\omega^{\hat{j}}{}_{\hat{i}} = \Gamma^j{}_i$, therefore (387a) and (387b) becomes

$$\boxed{R^j{}_{ikl} = b_{jk} b_{il} - b_{jl} b_{ik}} \tag{389}$$

and

$$\big(\partial_k b_{ij}\big) du^k \wedge du^j + \big(b_{jk} \Gamma^j{}_{il}\big) du^k \wedge du^l = 0 \qquad \Longrightarrow \qquad \boxed{\partial_k b_{ij} - \partial_j b_{ik} + b_{lk} \Gamma^l{}_{ij} - b_{lj} \Gamma^l{}_{ik} = 0\,,} \tag{390}$$

which have been given by (231) and (232) respectively.

Acknowledgments

Ling-Wei Luo (LWL) would like to thank Prof. Jen-Chi Lee (NCTU) who encurages LWL and gives an opportunity for LWL to give a lecture on elementary differential geometry for both undergraduate and dergraduate students in NCTU. LWL also would like to thans Mr. Shang-Yu Chien who is a PhD student in National Tsing Hua University to check and discuss the contents of this lecture note. The work was supported by Ministry of Science and Technology (MoST-107-2119-M-007-013-MY3) and Academia Sinica Career Development Award Program (AS-CDA-105-M06).

References

[1] 小林昭七: 曲線と曲面の微分幾何, 裳華房, 1995.

[2] 小林昭七: 接続の微分幾何とゲージ理論, 裳華房, 1989.

[3] 陳維桓: 微分幾何, 北京大學出版社, 2006.

[4] Élie Cartan, *On Manifolds with an Affine Connection and the Theory of General Relativity*, Bibliopolis, Napoli, Italy, 1986.

[5] Élie Cartan, *Riemannian Geometry in an Orthogonal Frame*, World Scientific Publishing Co., Singapore, 2001.

[6] S. S. Chern, Weihuan Chen, K. S. Lam, *Lectures on Differential Geometry*, World Scientific Publishing Co., Singapore, 1999.

[7] Manfredo P.do Carmo, *Differential Forms and Applications*, Springer, New York, 1998.

[8] Richard L. Faber, *Differential Geometry and Relativity Theory: An Introduction*, CRC Press, Boca Raton, FL, 1983.

[9] Harley Flanders, *Differential Forms with Applications to the Physical Sciences*, Dover Publications, New York, 1989.

[10] Shoshichi Kobayashi, Katsumi Nomizu, *Foundations of Differential Geometry, Vol. I*, John Wiley & Sons, New York, 1963.

[11] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, Freeman, SanFrancisco, 1973.

[12] J. A. Schouten *Ricci-Calculus: An Introduction to Tensor Analysis and Its Geometrical Applications*, Springer, Berlin, 1954.